Triply Regular Graphs

by

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Abstract

This project studies a regularity condition on graphs. A graph X is said to be *triply regular* if for $x, y, z \in V(X)$

$$|\{v \in V(X) : d(v,x) = i,\ d(v,y) = j \text{ and } d(v,z) = k\}|$$

is a constant depending only on $i, j, k \in \mathbb{Z}$ and the pairwise distances of $\{x, y, z\}$. We find a necessary condition for triply regular on strongly regular graphs; if a strongly regular graph meets the Krein bound, then it is triply regular.

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Chapter 1

Introduction

A graph on n vertices that is neither complete nor empty is *strongly regular* if each vertex has degree k, each pair of adjacent vertices has a common neighbours and each pair of non-adjacent vertices has c common neighbours. Let X be such a graph and consider a vertex x in X. Let

$$X_1(x) = \{y \in V(X) \,:\, y \sim x\}$$

and

$$X_2(x) = \{y \in V(X) \,:\, y \nsim x\}.$$

Since x has degree k, there are k vertices in $X_1(x)$. Each vertex in $X_1(x)$ is joined to a vertices in $X_1(x)$ and $k-a-1$ in $X_2(x)$. There are $n-k-1$ vertices in $X_2(x)$. Each vertex in $X_2(x)$ is joined to c vertices in $X_1(x)$ and so has $k-c$ neighbours in $X_2(x)$. See Figure 1. From a degree-counting argument, we obtain a fundamental equality satisfied by (n,k,a,c);

$$k(k-a-1) = c(n-k-1).$$

We also obtain that the subgraphs of X induced by $X_1(x)$ and $X_2(x)$ are both regular with valencies a and $k-c$, respectively.

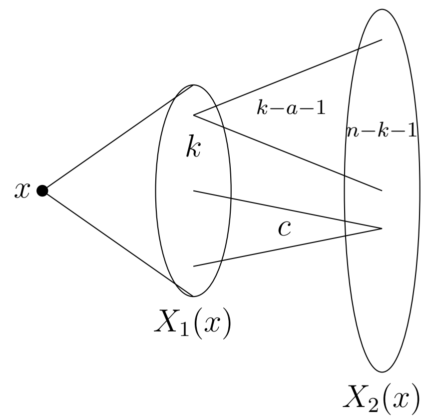

Figure 1.1: Neighbourhoods of a strongly regular graph with parameters (n, k, a, c).

It is natural to ask is when the graphs induced by $X_1(x)$ and $X_2(x)$ are strongly regular.

We come across the problem of strongly regular subconstituents while studying graphs with high regularity known as *triply regular graphs*, where for $x, y, z \in V(X)$,

$$|\{v \in V(X) \,:\, d(x,v) = i,\, d(y,v) = j,\, d(z,v) = k\}|$$

is a constant which only depends on i, j, k and ℓ, m, ν where

$$\ell = d(x,y),\; m = d(y,z),\; \text{and } \nu = d(z,x).$$

Triply regular graphs are discussed in [8, 11] in the context of spin models.

In this note, we study triply regular graphs from an algebraic graph theory perspective. In Chapter 2, we introduce basic concepts. In Chapter 3, we introduce two matrix algebras associated with a graph: the Bose-Mesner algebra and the Terwilliger algebra. In Chapter 4, we investigate strongly regular graphs that are triply regular. We show, following [9], that strongly regular graphs are triply regular if and only if they have strongly regular subconstituents. Then in Sections 4.3 and 4.4, we study strongly regular graphs with strongly regular subconstituents following a paper of Cameron, Goethals and Seidel [4]. We show that a strongly regular graph has strongly regular subconstituents with respect to any vertex when the Krein bound is met. We will also look briefly at classes of strongly regular graphs where there exist some vertex x with respect to which the subconstituents $X_1(x)$ and $X_2(x)$ are strongly regular. Finally, we will pose some open questions concerning triply regular graphs.

Chapter 2

Preliminaries

2.1 Notation

For a graph X, we denote by $X_i(x)$ the ith neighbourhood of x; that is

$$X_i(x) = \{y \in V(G) : d(x, y) = i\}.$$

The subgraph induced by X_i is called the ith *subconstituent* of X.

2.2 Distance regular graphs

A *distance regular graph* is a graph X with diameter d such that there exists $\{b_i\}_{i=0}^{d-1}$ and $\{c_i\}_{i=1}^{d}$ such that for $x, y \in V(X)$ with $d(x, y) = i$,

$$b_i = |X_1(x) \cap X_{i+1}(y)|$$

and

$$c_i = |X_1(x) \cap X_{i-1}(y)|.$$

The array $\{b_0, \ldots, b_{d-1}\,; c_1, \ldots, c_d\}$ is said to be the *intersection array* of X. One may observe that X must be regular with valency b_0 and c_1 is always equal to 1. If we consider $x, y \in V(X)$ at distance i apart, then the number of neighbours of x in $X_i(y)$ is $a_i = k - b_i - c_i$.

For any graph G, a partition π of the vertices of G with cells $V_1, \ldots, V_m$ is said to be *equitable* if the number of neighbours of $v \in V_i$ in V_j is a constant b_{ij} which does not depend on the choice of v. For a distance regular graph X, we can see that the *distance partition* $\{\{x\}, X_1(x), \ldots, X_d(x)\}$ for any

$x \in V(X)$ is an equitable partition of X. The converse is also true and can be found in [3].

2.2.1 Theorem. *A graph X is distance regular if and only its distance partition is equitable with the same intersection array, with respect to any vertex x of X.*

It is worth noting that the distance regular condition is equivalent to requiring for any two vertices x and y of X at distance k, that the number of vertices at distance i from x and distance j from y is a constant p_{ij}^k which does not depend on the choice of x and y.

2.3 Strongly regular graphs

A *strongly regular graph* is a graph X on n vertices that is neither complete nor empty where each vertex has degree k, each pair of adjacent vertices has a common neighbours and each pair of non-adjacent vertices has c common neighbours. A connected strongly regular graph is a distance regular graph of diameter 2 and has intersection array $\{k, k-a-1; 1, c\}$. We refer to the tuple (n, k, a, c) as the parameter of X.

The following theorem is a standard result relating the property of being strongly regular to the eigenvalues of the graph. We follow the proof of [6] for the first direction and give a slightly modified proof for the second direction.

2.3.1 Theorem. *A connected regular graph X is strongly regular if and only if it has exactly 3 distinct eigenvalues.*

Proof. First we will show for that X, a connected strongly regular graph with parameters (n, k, a, c), has exactly three eigenvalues. Let A be the adjacency matrix of X. The uvth entry of A^2 is the number of walks of length 2 from u to v. Since X is strongly regular, the uvth entry of A^2 depends only on the adjacency of u and v; that is

$$(A^2)_{uv} = \begin{cases} k & \text{if } u = v \\ a & \text{if } u \sim v \\ c & \text{if } u \nsim v. \end{cases}$$

Then, the adjacency matrix of X satisfies the following equation.

$$A^2 + (c-a)A + (c-k)I - cJ = 0 \tag{2.3.1}$$

where I is the the $n \times n$ identity matrix and J is the $n \times n$ all ones matrix.

Since X is regular, k is an eigenvalue of X with the all ones vector as the corresponding eigenvector. Since A is a real symmetric matrix, we can find an orthonormal eigenbasis of $\mathbb{R}^n$ with respect to A, say $\mathbf{v}_1, \ldots, \mathbf{v}_n$, with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. We may assume that $\lambda_1 = k$ and $\mathbf{v}_1$ is the all ones vector. Consider $\mathbf{v}_i$, where $i > 1$. Applying $\mathbf{v}_i$ to 2.3.1, we obtain

$$\begin{aligned}(A^2 + (c-a)A + (c-k)I - cJ)\mathbf{v}_i &= 0\\ A^2\mathbf{v}_i + (c-a)A\mathbf{v}_i + (c-k)I\mathbf{v}_i - cJ\mathbf{v}_i &= 0\\ \lambda_i^2\mathbf{v} + (c-a)\lambda_i\mathbf{v} + (c-k)\mathbf{v} &= 0.\end{aligned}$$

Then, every $\lambda \in \{\lambda_i\}_{i=2}^n$ must satisfy

$$\lambda^2 + (c-a)\lambda + (c-k) = 0. \tag{2.3.2}$$

This is the minimal polynomial of A over the orthogonal complement of the all ones vector in $\mathbb{R}^n$. If we let $\Delta = (a-c)^2 + 4(k-c)$ be the discriminant, the two roots of 2.3.2 are

$$\theta = \frac{(a-c) + \sqrt{\Delta}}{2},$$

and

$$\tau = \frac{(a-c) - \sqrt{\Delta}}{2}.$$

Further, we can find that the multiplicities of the eigenvalues in terms of the parameters. Let m_θ and m_τ be the multiplicities of θ and τ respectively. We have that

$$1 + m_\theta + m_\tau = n$$

and

$$l + \theta m_\theta + \tau m_\tau = 0.$$

From these equations it is clear that θ, τ, m_θ and m_τ depend only on the parameters of X.

Conversely, suppose X is a connected regular graph with adjacency matrix A with three eigenvalues k, θ and τ, where k is the valency. Then, the characteristic polynomial of A is

$$p(A) = (A - kI)(A - \theta I)(A - \tau I).$$

Since X is connected and regular, the Perron-Frobenius theorem gives that k has multiplicity 1. Let $\mathbf{1}$ be the all ones vector and consider all vectors in the orthogonal complement W of $\mathbf{1}$ in $\mathbb{R}^n$. The minimal polynomial of A over W is

$$q(A) = (A - \theta I)(A - \tau I)$$

and vanishes on W. If we apply $\mathbf{1}$ to $q(A)$ we get

$$\begin{aligned} q(A) &= (A - \theta I)(A - \tau I)\mathbf{1} \\ &= (A - \theta I)(A\mathbf{1} - \tau I\mathbf{1}) \\ &= (A - \theta I)(k\mathbf{1} - \tau\mathbf{1}) \\ &= (k - \tau)(A - \theta I)\mathbf{1} \\ &= (k - \tau)(A\mathbf{1} - \theta I\mathbf{1}) \\ &= (k - \tau)(k\mathbf{1} - \theta\mathbf{1}) \\ &= (k - \tau)(k - \theta)\mathbf{1}. \end{aligned}$$

We consider the matrix polynomial,

$$r(A) = (A - \theta I)(A - \tau I) - \frac{(k - \tau)(k - \theta)}{n} J.$$

If $\mathbf{y} \in W$, then $J\mathbf{y} = 0$, so $r(A)$ vanishes under multiplication by vectors in W. By the above calculation we see that

$$r(A)\mathbf{1} = (k - \tau)(k - \theta)\mathbf{1} - \frac{(k - \tau)(k - \theta)}{n} n\mathbf{1} = 0.$$

Then $r(A)$ vanishes under right multiplication by all vectors in $\mathbb{R}^n$ and is hence the 0 matrix. We have found a polynomial in A, I and J with degree 2 in A and this suffices to show that X is strongly regular. $\square$

A strongly regular graph X is said to be *primitive* if both X and its complement are connected. Otherwise it is said to be *imprimitive*. We offer this standard lemma from [6] , which will be useful, without proof.

2.3.2 Lemma. *If X is a strongly regular graph with parameter (n, k, a, c) then the following are equivalent*

(a) X is not connected,

(b) $c = 0$,

(c) $a = k - 1$ *and*

(d) X *is isomorphic to the disjoint union of more than one copy of* K_{k+1}.

Lemma 2.3.2 gives us that the only imprimitive graphs are the disjoint unions of complete graphs and the complements of such graphs.

2.4 Regularity conditions

A graph X is said to be *triply regular* if for $x, y, z \in V(X)$ the number

$$|X_i(x) \cap X_j(y) \cap X_\ell(z)|$$

for any choice of i, j and ℓ depends only the distances $d(x, y)$, $d(y, z)$ and $d(z, x)$.

It is clear that if a graph is triply regular, then it must be distance regular and so we may view triply regular as a generalization of distance regular.

Chapter 3

Matrix Algebras

Let X be a graph with diameter d and let A be its adjacency matrix. We can consider the matrix subalgebra of $M_{n\times n}(\mathbb{R})$ generated by I, J and the powers of A. If X is strongly regular, we saw that A^2 is a linear combination of A, I and J and so this matrix algebra has dimension 3. The following theorem on distance regular graphs is standard in the literature. The proof follows from the intersection array of X and will not be included here.

3.0.1 Theorem. *[1] The graph X is a distance regular graph of diameter d if and only if the matrix algebra generated by I, J and the powers of A has dimension $d+1$.*

From Theorem 3.0.1, we see that a distance regular graph has at most $d+1$ distinct eigenvalues. Any graph with diameter d has at least $d+1$ distinct eigenvalues, so a distance regular graph of diameter d must have exactly $d+1$ eigenvalues. Observe that A^{ℓ}_{uv} is the number of walks of length ℓ from u to v in X. Let A_i be the matrix indexed by the vertices of X such that the uvth entry is 1 if $d(u,v) = i$ and 0 otherwise. The *Bose-Mesner algebra of X* is the algebra generated by $A_0, A_1, \dots A_d$. Since A^i can be expressed as a linear combination of A_j for $j < i$, we can rewrite Theorem 3.0.1 as follows:

3.0.2 Theorem. *The graph X is a distance regular graph of diameter d if and only if the Bose-Mesner algebra of X has dimension $d+1$.*

In the next section, we will look at basic properties of the Bose-Mesner algebra of a distance regular graph.

3.1 Bose-Mesner algebra

For this section, we will let X be a distance regular graph with diameter d. Let A be the adjacency matrix of X and let

$$\mathcal{A} = \{A_0, A_1, \ldots, A_d\}$$

be the distance matrices of X. Observe that $A_1 = A$ and A_i is symmetric for all i. The uv entry of A_iA_j is the number of vertices at distance i from u and j from v, which does not depend on the choice of u and v, since X is distance regular. Then A_iA_j is symmetric and

$$A_iA_j = A_i^TA_j^T = (A_jA_i)^T = A_jA_i$$

which is to say that the Bose-Mesner algebra of A is a commutative algebra. The following are true for $\mathcal{A}$:

(i) $A_0 = I$,

(ii) $\sum_{i=0}^{d} A_i = J$,

(iii) $A_i^T \in \mathcal{A}$ for each i,

(iv) $A_iA_j = A_jA_i$ is the the span of $\mathcal{A}$ and

(v) $A_i \circ A_j = 0$

where $\circ$ denotes the entry-wise matrix product (called the *Schur product* or the *Hadamard product).*

In general, a set of 01-matrices which satisfies a), b), c) and d) is said to be an *association scheme*. We see that the Bose-Mesner algebra of any distance regular graph is an association scheme, but the converse is not true. An association scheme whose matrices are the distance matrices of a distance regular graph is said to be a *metric association scheme*. An important property of metric association scheme is that A_j is a linear combination of the powers of A.

For each i, let $p_i(t)$ be the polynomial such that

$$p_i(A) = A_i.$$

Let λ be an eigenvalue of A with eigenvector $\mathbf{v}$. We have that

$$A_i\mathbf{v} = p_i(A)\mathbf{v} = p_i(\lambda)\mathbf{v}$$

and so $\mathbf{v}$ is an eigenvector of A_i with eigenvalue $p_i(\lambda)$ for each i. Then the matrices of $\mathcal{A}$ have the same eigenvectors. This gives us hope that the simultaneous eigenspaces of the A_is will give us more information about the Bose-Mesner algebra. Indeed, we will be able to construct another matrix basis of the span of $\mathcal{A}$ using the eigenspaces of A.

Let $\lambda_0, \ldots, \lambda_d$ be the eigenvalues of A with eigenspaces $W_0, \ldots, W_d$, respectively. For each $i \in \{0, \ldots, d\}$ let U_i be the matrix whose columns form an orthonormal basis for W_i and let

$$E_i = U_i U_i^T$$

be the projection onto W_i. Since X is regular, we may assume without loss of generality that λ_0 is the valency of X. We have the following lemma.

3.1.1 Lemma. *Let X, $\mathcal{A}$ and $\{E_i, \lambda_i, p_i(t)\}_{i=0}^d$ be as defined above. Then for each $i, j \in \{0, \ldots, d\}$ the following are true:*

(a) $E_0 = cJ$ *for some constant* $c \in \mathbb{R}$,

(b) $\displaystyle\sum_{i=0}^{d} E_i = I$,

(c) *span*$(\{E_i\}_{i=0}^d) =$ *span*$(\mathcal{A})$, *and*

(d) $E_i E_j = \delta_{ij} E_i$.

where δ denotes the Kronecker delta.

Proof. Part a) follows from the assumption that λ_0 is the valency of the graph since W_0 is spanned by the all-ones vector. By definition of the U_i's, we have that

$$E_i^2 = U_i U_i^T U_i U_i^T = E_I$$

and for $i \neq j$

$$E_i E_j = U_i U_i^T U_j U_j^T = 0$$

since the vectors in different eigenspaces of a symmetric matrix are orthogonal. We see that part d) is true.

Since A is a symmetric matrix, its eigenspaces form an orthogonal partition of $\mathbb{R}^n$ and so we can decompose any vector $\mathbf{v} \in \mathbb{R}^n$ into the sum of its projections on the eigenspaces of A. That is to say that for $\mathbf{v} \in \mathbb{R}^n$,

$$\mathbf{v} = E_0\mathbf{v} + \cdots + E_d\mathbf{v}.$$

Since this is true for all vectors in $\mathbb{R}^n$, we have that

$$E_0 + \cdots + E_d = I$$

and part b) follows.

To show that part c), we will first show that each A_i is in the span of the E_i's. Then, the statement of part c) will follow from the fact that $\mathrm{span}(\mathcal{A}) \subseteq \mathrm{span}(\{E_i\}_{i=0}^d)$ and their bases have the same cardinality. In particular, we will show

$$A_i = \sum_{j=0}^{d} p_i(\lambda_j)E_j.$$

Consider A_i and E_j. Each column of U_j is an eigenvalue of A_i with eigenvalue $p_i(\lambda_j)$ so,

$$\begin{aligned} A_iE_j &= A_iU_jU_j^T \\ &= (A_iU_j)U_j^T \\ &= (p_i(\lambda_j)U_j)\,U_j^T \\ &= p_i(\lambda_j)E_j. \end{aligned}$$

Now we use the projection of the columns of A onto the partition of $\mathbb{R}^n$ imposed by the E_i's to obtain

$$\begin{aligned} A_i &= A_iE_0 + A_iE_1 + \cdots + A_iE_d \\ &= p_i(\lambda_0)E_0 + p_i(\lambda_1)E_1 + \cdots + p_i(\lambda_d)E_d \end{aligned}$$

as required. □

From Lemma 3.1.1, we see that $\{E_i\}_{i=0}^d$ is another basis for the Bose-Mesner algebra of X which behaves nicely under the matrix product. Observe that property e) of $\mathcal{A}$ implies that the Bose-Mesner algebra is closed under the Schur product. It follows that $E_i \circ E_j \in \mathrm{span}(\mathcal{A})$. In particular, there exists constant q_{ij}^k such that

$$E_i \circ E_j = \sum_{k=0}^{d} q_{ij}^k E_k.$$

The constants q_{ij}^k, where i, j and k range from 0 to d, are called the *Krein parameters of* X. We always have that $q_{ij}^k \geq 0$ for all $0 \leq i, j, k \leq d$ since they are eigenvalues of the Schur product of two positive semidefinite matrices and are hence nonnegative. In the literature, these inequalities is refered to as the *Krein bounds*. In Section 4.2, we will give a more complicated proof for the Krein bounds of strongly regular graphs and obtain information about the graph when one of the Krein bounds is tight.

3.2 Terwilliger algebra

We defined the Bose-Mesner algebra of a distance regular graph X with diameter d to be the matrix algebra generated by the distance matrices $\mathcal{A} = \{A_0, \ldots, A_d\}$ of X. To study triply regular graphs, we will need a matrix algebra which captures more information about the distance classes of X. To this end, we will define the Terwilliger algebra of X.

Fix a vertex $x \in V(X)$ and recall that $X_i(x)$ denote the set of vertices at distance i from x. Let F_i be a diagonal matrix such that the (u, u) position of F_i is 1 if $d(u, x) = i$ and 0 otherwise. Essentially, the F_i's are diagonal matrices with the characteristic vectors of the distance partition of X with respect to x on the diagonal. The matrix algebra generated by the distance matrices $\mathcal{A}$ and the diagonal matrices $F_0, \ldots, F_d$ is called the *Terwilliger algebra of* X *with respect to* x. We will call x the *vertex of origin*.

We established in Section 2.2 that the distance partition of a distance regular graph with respect to any vertex will be equitable. The Terwilliger algebra of X may be different depending on the choice of the vertex of origin. The Terwilliger algebra was first introducted by Terwilliger in 1992 in [12].

We will now show some basic, technical properties of the Terwilliger algebra.

From the definition of F_i for $i \in \{0, \ldots, d\}$, we see that

$$I = A_0 = \sum_{i=0}^{d} F_i.$$

For $0 \leq i, j \leq d$, both F_i and F_j are diagonal matrices with the characteristic vectors of a partition of V as diagonal, so we have that

$$F_i F_j = \delta_{ij} F_i.$$

Due to the combinatorial interpretation of the A_i's and F_j's, some products can be expressed easily. We can consider the three-fold product

$$F_i A_j F_k$$

for $0 \leq i, j, k \leq d$. We can find the (u, v) entry of this product by direct computation:

$$(F_i A_j F_k)_{uv} = \begin{cases} 1 & \text{if } u \in X_i(x),\ v \in X_k(x) \text{ and } u \in X_j(v) \\ 0 & \text{otherwise.} \end{cases} \tag{3.2.1}$$

Using equation 3.2.1, we can find that

$$F_0 A_j = F_0 A_j F_j$$

and

$$A_j E_0 = E_j A_j E_0.$$

By more direct calculation, we can see that

$$(A_i F_j)_{uv} = \begin{cases} 1 & \text{if } d(u, v) = i \text{ and } d(v, x) = j \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$\begin{aligned} (A_i F_j A_k)_{uw} &= \sum_{v \in V(X)} (A_i F_j)_{uv} (A_k)_{vw} \\ &= \sum_{v \in X_k(w)} (A_i F_j)_{uv} \\ &= |\{v \in X_k(x) \,:\, v \in X_i(u) \cap X_j(x)\}| \\ &= |X_i(u) \cap X_j(x) \cap X_k(w)| \end{aligned}$$

and we have

$$(A_i F_j A_k)_{uw} = |X_i(u) \cap X_j(x) \cap X_k(w)| \tag{3.2.2}$$

for any $0 \leq i, j, k \leq d$. Using equation 3.2.2, we can calculate further that

$$A_i F_0 A_k = F_i J F_k.$$

Since $\mathcal{A}$ is an association, we know there exists constants p_{ij}^k such that

$$A_iA_j = \sum_{k=0}^{d} p_{ij}^k A_k.$$

Using the p_{ij}^k and the above calculations, we can simplify some four-fold products of the A_i's and F_j's, as follows:

$$F_0A_iF_jA_k = \delta_{ij} \sum_{\ell=0}^{d} p_{ik}^{\ell} F_0A_\ell F_\ell$$

and

$$A_iF_jA_kE_0 = \delta_{jk} \sum_{\ell=0}^{d} p_{ik}^{\ell} F_\ell A_\ell F_0.$$

The proof follows from direct calculation and are not included here.

We summarize these technical properties and some immediate consequences in the following lemma.

3.2.1 Lemma. *The following are true for the Terwilliger algebra of a distance regular graph:*

(i) $I = A_0 = \sum_{i=0}^{d} F_i,$

(ii) $A_iA_j = \sum_{k=0}^{d} p_{ij}^k A_k,$

(iii) $F_iF_j = \delta_{ij}F_i,$

(iv)

$$(F_iA_jF_k)_{uv} = \begin{cases} 1 & \text{if } u \in X_i(x),\ v \in X_k(x) \text{ and } u \in X_j(v) \\ 0 & \text{otherwise,} \end{cases}$$

(v) $F_0A_j = F_0A_jF_j$ *and* $A_jE_0 = E_jA_jE_0,$

(vi) $(A_iF_jA_k)_{uw} = |X_i(u) \cap X_j(x) \cap X_k(w)|,$

(vii) $A_iF_0A_k = F_iJF_k,$

(viii) $JF_jA_k = \sum_{i=0}^{d} p^i_{jk} JF_i$,

(ix) $A_iF_jJ = \sum_{k=0}^{d} p^k_{ji} F_kJ$,

(x) $F_0A_iF_jA_k = \delta_{ij} \sum_{\ell=0}^{d} p^\ell_{ik} F_0A_\ell F_\ell$, and

(xi) $A_iF_jA_kE_0 = \delta_{jk} \sum_{\ell=0}^{d} p^\ell_{ik} F_\ell A_\ell F_0$.

Chapter 4

Strongly regular triply regular graphs

In Section 4.1, we will use the Terwilliger algebra of the graph to show the following theorem of Munemasa [9] describing triply regular strongly regular graphs.

4.0.2 Theorem. *A strongly regular graph is triply regular if and only if its subconstituents with respect to any vertex are strongly regular.*

We will then proceed to study strongly regular graphs with strongly regular subconstituents. For a graph to have strongly regular subconstituents, it suffices to have one tight Krein parameter [4]. There are two approaches to the proof. Godsil and Royle approach the problem by studying the local eigenvalues of the graph as found in [6], which we will visit briefly in Section 4.2. In Section 4.3, we will look at the original proof of Cameron, Goethals and Seidel in [4], which uses the spherical t-designs formed by the projections of some orthonormal basis of $\mathbb{R}^n$ associated with the vertices of a strongly regular graph X onto the eigenspaces of X. In Section 4.4 we will show a theorem about necessity of strongly regular graphs with strongly regular subconstituents. Finally, in Section 4.5, we will look at small examples of graph with strongly regular subconstituents.

4.1 Strongly regular graphs

In this section, we will prove Theorem 4.0.2 following [9]. In order to prove this, we need a few properties of the Terwilliger algebra. Let X be a strongly regular graph and fix vertex $x \in V(X)$. Let A_0, A_1 and A_2 be the distance matrices of X. For $i = 0, 1, 2$, let F_i be the diagonal matrix with the characteristic vector of $X_i(x)$. We will consider T the Terwilliger algebra of X with respect to vertex x.

Let T_0 be the linear subspace of T spanned by

$$\{F_i A_j F_k \,:\, 0 \leq i, j, k \leq d\}.$$

Clearly, T_0 contains A_i for every i and F_j for every j. Then T_0 generates T as an algebra but may be a proper subspace of T.

4.1.1 Lemma. *The subspace T_0 is closed under the Schur product.*

Proof. It is easy to see by Lemma 3.2.1 part (iv) that

$$F_\ell \circ (F_i A_j F_k) = F_\ell$$

when $i = k = \ell$ and $j = 0$ and

$$F_\ell \circ (F_i A_j F_k) = 0$$

otherwise. Following some calculations using Lemma 3.2.1, we also obtain

$$A_\ell \circ (F_i A_j F_k) \in T_0$$

and the lemma follow. □

4.1.2 Lemma. *A distance regular graph X is triply regular if and only if $T = T_0$ for every choice of the vertex of origin.*

Proof. Recall that X is triply regular if

$$|X_i(u) \cap X_j(v) \cap X_k(w)|$$

is independent of the choice of u, v and w, such that the pairwise distances are $d(u, v) = m_1$, $d(v, w) = m_2$ and $d(w, u) = m_3$. From Lemma 3.2.1 part (vi), we have that

$$(A_i F_j A_k)_{uw} = |X_i(u) \cap X_j(x) \cap X_k(w)|,$$

where x is the vertex of origin of T. Then Lemma 3.2.1 part (vi), X is triply regular if and only if $A_i F_j A_k$ is a linear combination of scalar multiples of matrices of form $F_\ell A_m F_\nu$, for all $0 \leq i, j, k \leq d$. Then X is triply regular if and only if $A_i F_j A_k \in T_0$ for all $0 \leq i, j, k \leq d$. If $T_0 = T$, then we have that X is triply regular.

For the converse, suppose that X is triply regular. We will consider elements of T as linear combinations of words in A_i and F_j. We will show that $M \in T$ is in T_0 by induction on the maximum length of the words in M, denoted $s(M)$. Clearly since A_i and F_j are in T_0 for all i and j, the base case is true. Consider M with $s(M) > 1$. If in the longest word of M, we have consecutive matrices of the same type, i.e. $A_i A_j$ or $F_i F_j$ for some i and j, then we can reduce $s(M)$ using Lemma 3.2.1 part (ii) or part (iii). Otherwise, the longest word of M alternates between A_i's and F_j's and we can reduce its length using the above observation that $A_i F_j A_k$ is a linear combination of scalar multiples of matrices of form $F_\ell A_m F_\nu$, for all $0 \leq i, j, k \leq d$. Then, $M \in T_0$ by induction. □

We assume from this point forward, that we must prove the statement for any choice of x the vertex of origin. If X has diameter 2, then we can simplify the requirement for $T_0 = T$.

4.1.3 Lemma. *A strongly regular graph X is triply regular if and only if $A_1 F_1 A_1 \in T_0$.*

The proof of Lemma 4.1.3 follows easily by relating $A_i F_j A_k$ to $A_1 F_1 A_1$ using Lemma 3.2.1 and will be omitted here. Now we are ready to prove Theorem 4.0.2 in two steps.

4.1.4 Theorem. *If X is a strongly regular graph, then X is triply regular if and only if its subconstituents with respect to any vertex are distance regular.*

Proof. Observe that $F_i A_i F_i$ is a block matrix with the adjacency matrix of the ith subconstituent of X as a principal block and 0 elsewhere. Then, if X is triply regular, X induces an association scheme on the subconstituents of X.

Now suppose that X induces an association scheme on the subconstituents of X. Then

$$(F_1 A_1 F_1)^2 = F_1 A_1 F_1 F_1 A_1 F_1 = F_1 A_1 F_1 A_1 F_1$$

and

$$(F_2 A_1 F_2)^2 = F_2 A_1 F_2 F_1 A_2 F_2 = F_2 A_1 F_2 A_1 F_2$$

are in the spans of $\{F_1A_1F_1, F_1A_0F_1, F_1JF_1\}$ and $\{F_2A_1F_2, F_2A_0F_2, F_2JF_2\}$, respectively, and so are in T_0. Then $F_2A_1F_1A_1F_2 \in T_0$. By Lemma 4.1.1, we have that

$$A_2 \circ F_1A_1F_1A_1F_1$$

and

$$A_1 \circ F_2A_1F_1A_1F_2$$

are both in T_0. We can observe that the following are equivalent:

- $A_2 \circ F_1A_1F_1A_1F_1 \in T_0$ and
- $A_1 \circ F_2A_1F_1A_1F_1 \in T_0$.

This follows from the technical observation that both of these statements is equivalent to the condition that

$$|X_1(w) \cap X_1(y) \cap X_1(z)|$$

is a constant independent of the choice of w, y, z such that $d(w, y) = 1$, $d(y, z) = 1$ and $d(w, z) = 2$. Similarly, the following are equivalent:

- $A_1 \circ F_2A_1F_1A_1F_2 \in T_0$ and
- $A_2 \circ F_2A_1F_1A_1F_2 \in T_0$.

This follows from the technical observation that both of these statements is equivalent to the condition that

$$|X_1(x) \cap X_1(y) \cap X_1(z)|$$

is a constant independent of the choice of x, y, z such that $d(x, y) = 2$, $d(y, z) = 1$ and $d(x, z) = 2$.

Then we have that

$$A_1F_1A_1 = (F_0 + F_1 + F_2)A_1F_1A_1(F_0 + F_1 + F_2)$$

so $A_1F_1A_1 \in T_0$ and X is triply regular as required. □

4.1.5 Lemma. *If the subconstituents of a strongly regular graph are distance regular then they are also strongly regular.*

Proof. Suppose X is strongly regular and triply regular and the ith subconstituent of X with respect to vertex x is distance regular of diameter $e > 2$ for some $i \in \{1, 2\}$ and some $x \in V(X)$. Let $\{b_0, \ldots, b_{e-1}; c_1, \ldots, c_e\}$ be the intersection array of the ith subconstituent of X. Consider $u, v, w, y \in X_i(x)$ such that vertices u and v are not adjacent in X, vertices w and y are not adjacent in X and the distances of the pairs $\{u, v\}$ and $\{w, x\}$ are different in the ith subconstituent of X. Since X is triply regular, we obtain

$$|X_i(x) \cap X_1(u) \cap X_j(v)| = |X_i(x) \cap X_1(x) \cap X_j(y)|.$$

This implies that $b_2 = b_3 = \cdots = b_{e-1}$ and $c_2 = \cdots = c_e$, which contradictions feasibility condition for intersection arrays of distance regular graph. □

We see that Theorem 4.1.4 together with Lemma 4.1.5 imply Theorem 4.0.2.

4.2 The Krein bounds

Let X be a strongly regular graph. We consider the first and second subconstituent of X with respect to some vertex x and we will abuse notation and use $X_i(x)$ to denote the graph induced by the vertices at distance i from X. An eigenvalue of $X_i(x)$ is said to be a *local eigenvalue* if it is not an eigenvalue of X. By studying local eigenvalue and using Theorem 2.3.1, Godsil and Royle show the following theorem. The details can be found in [6, p. 227-235].

4.2.1 Theorem. *Let X be a primitive strongly regular graph with parameters (n, k, a, c) and eigenvalues k, θ and τ, with multiplicities, 1, m_θ and m_τ respectively. Then,*

$$q_{11}^1 = \theta\tau^2 - 2\theta^2\tau - \theta^2 - k\theta + k\tau^2 + 2k\tau \geq 0$$

and

$$q_{22}^2 = \theta^2\tau - 2\theta\tau^2 - \theta^2 - k\tau + k\theta^2 + 2k\theta \geq 0.$$

If either inequality is tight, then one of the following is true:

(i) X is the 5-cycle,

(ii) either X or its complement has all its first subconstituents empty, and all its second subconstituents strongly regular, or

(iii) all subconstituents of X are strongly regular.

4.3 Spherical t-designs

Let X be a strongly regular graph with parameters (n, k, a, c). Let $A_0 = I, A_1, A_2$ be the distance matrices of X. Let $\theta \geq \tau$ be the eigenvalues of X, other than k and let their multiplicities be m_θ and m_τ respectively. Let W_0, W_1, W_2 be the eigenspaces of X, corresponding to k, θ, and τ respectively. Let E_i be the orthogonal projections onto the W_i for $i = 0, 1, 2$, as in Chapter 3. We identify the vertices of X with an orthonormal basis $\mathcal{X} = \{x_1, \ldots, x_n\}$ of $V := \mathbb{R}^n$. We see that

$$V = W_0 \oplus W_1 \oplus W_2.$$

Let Y_i be the set of projections of $\{x_j\}_{j=1}^n$ onto W_i, which is to say that

$$Y_i = \{E_i x_j \, : \, j = 1, \ldots, n\}.$$

Recall that $E_i = U_i U_i^T$ where U_i is the matrix with an orthonormal basis $\mathcal{B}_i$ of W_i as columns. We always have that $U_0 = \frac{1}{\sqrt{n}}\mathbf{1}_n$ where $\mathbf{1}_m$ is the all ones vector of dimension m. Then,

$$U = [U_0 \quad U_1 \quad U_2]$$

is the orthogonal transition matrix from $\mathcal{X}$ to $\mathcal{B} = \mathcal{B}_0 + \mathcal{B}_1 + \mathcal{B}_2$.

Using standard matrix arguments, we know that there exists constants α_i and β_i such that

$$\frac{n}{m_\theta} E_1 = I + \alpha_1 A_1 + \beta_1 A_2$$

and

$$\frac{n}{m_\tau} E_2 = I + \alpha_2 A_1 + \beta_2 A_2.$$

Then, we may assume the first basis vector of U_i is

$$\begin{pmatrix} 1 \\ \alpha_i \mathbf{1}_k \\ \beta_i \mathbf{1}_\ell \end{pmatrix}$$

where $\ell = n - k - 1$. Let $\mathbf{0}_m$ be the zero vector of dimension m. Then, we can write U into blocks as

$$U = \left(\begin{array}{c|c|c|c|c}
\frac{1}{\sqrt{n}} & \sqrt{\frac{m_\theta}{n}} & \mathbf{0}_{m_\theta}^T & \sqrt{\frac{m_\tau}{n}} & \mathbf{0}_{m_\tau}^T \\
\hline
\frac{1}{\sqrt{n}}\mathbf{1}_k & \sqrt{\frac{m_\theta}{n}}\alpha_1\mathbf{1}_k & \frac{1}{\sqrt{\theta-\tau}}K_1 & \sqrt{\frac{m_\tau}{n}}\alpha_2\mathbf{1}_k & \frac{1}{\sqrt{\theta-\tau}}K_2 \\
\hline
\frac{1}{\sqrt{n}}\mathbf{1}_\ell & \sqrt{\frac{m_\theta}{n}}\beta_2\mathbf{1}_\ell & \frac{1}{\sqrt{\theta-\tau}}L_1 & \sqrt{\frac{m_\tau}{n}}\beta_2\mathbf{1}_\ell & \frac{1}{\sqrt{\theta-\tau}}L_2
\end{array}\right)$$

where K_1, K_2, L_1 and L_2 are $k \times (m_\theta - 1)$, $k \times (m_\tau - 1)$, $\ell \times (m_\theta - 1)$ and $\ell \times (m_\tau - 1)$ matrices, respectively. Then, since the columns of U are an orthonormal basis, we have

$$U^T U = I$$

and be obtain the following lemma.

4.3.1 Lemma. *For $i = 1, 2$,*

$$K_i^T \mathbf{1}_k = 0, \quad L_i^T \mathbf{1}_\ell = 0,$$

and

$$K_i^T K_i + L_i^T L_i = (\theta - \tau)I.$$

Recall that the Krein parameters q_{ij}^k are given by

$$E_i \circ E_j = \sum_{k=0}^{2} q_{ij}^k E_k$$

for $i, j \in \{0, 1, 2\}$. We need the following technical lemma, whose proof we will omit.

4.3.2 Lemma. *The matrice $\Phi_i := \frac{1}{\sqrt{\theta-\tau}}(\alpha_i K_i^T K_i + \beta_i L_i^T L_i)$ for $i = 1, 2$ satisfies*

$$(m_\theta - 1)tr(\Phi_1^2) - tr(\Phi_1)^2 = \frac{q_{11}^1(m_\theta - 1 - nq_{11}^1)n}{m_\theta}$$

and

$$(m_\tau - 1)tr(\Phi_2^2) - tr(\Phi_2)^2 = \frac{q_{22}^2(m_\tau - 1 - nq_{22}^2)n}{m_\tau}.$$

If $\Phi_i = \gamma I$ for some γ, then $q_{ii}^i = 0$ or $(\mu_i - 1)$ divides n, where $\mu_1 = m_\theta$ and $\mu_2 = m_\tau$. Further, if $q_{ii}^i = 0$, then $\Phi_i = 0$.

A *spherical t-design* is a finite subset S of the unit sphere in $\mathbb{R}^m$ such that the average value of any polynomial of degree at most t in m variables over the points in S equals its average value of the sphere. We can find that S is a spherical 1-design if and only if

$$\sum_{s \in S} s = 0.$$

Let Z be a matrix such that the columns of Z are the elements of S. Then S is a 2-design if and only if ZZ^T is a nonzero multiple of the identity matrix. Any spherical 2-design which is a 2-distance set gives a pair of strongly regular graphs. A description of spherical designs which contains the proofs of the above statements can be found in [7, p. 272-273].

Fix $x \in V(X)$ and let x_j be the basis vector associated with x. Let Ω_i be the subspace of V_i orthogonal to $E_i x_j$.

4.3.3 Theorem. *For $i = 1, 2$, the projection of $X_1(x)$ and $X_2(x)$ into the subspace Ω_i are spherical 2-designs of k and ℓ, respectively, if and only if $q_{ii}^i = 0$.*

Proof. By definition, K_i and L_i are matrices with the vectors of the projection of $X_1(x)$ and $X_2(x)$ into the subspace Ω_i as columns, respectively. Then, Lemma 4.3.1 gives that the projections are 1-designs. Then Lemmas 4.3.1 and 4.3.2 give that

$$\alpha_i K_i^T K_i + \beta_i L_i^T L_i = 0$$

and

$$K_i^T K_i + L_i^T L_i = (\theta - \tau) I.$$

Hence $K_i^T K_i$ and $L_i^T L_i$ are nonzero multiples of the identity matrix. Then, $K_i^T K_i$ and $L_i^T L_i$ are the Gram matrices of spherical 2-designs in Ω_i. Conversely, if $K_i^T K_i$ and $L_i^T L_i$ are the Gram matrices of spherical 2-designs in W_i, then $\Phi_i = \gamma I$ and $q_{ii}^i = 0$. □

We know that the vectors of Y_i form a spherical 2-distance set in V_i from

$$\frac{n}{m_\theta} E_1 = I + \alpha_1 A_1 + \beta_1 A_2$$

and

$$\frac{n}{m_\tau} E_2 = I + \alpha_2 A_1 + \beta_2 A_2.$$

Then we have shown the following theorem.

4.3.4 Theorem. *[4] Let X be a strongly regular graph. Suppose $q_i i^i = 0$ for some $i \in \{1, 2\}$. Then, for every vertex x, the subconstituents $X_1(x)$ and $X_2(x)$ are both strongly regular.*

4.4 Smith graphs

Let X be a strongly regular graph with eigenvalues $k \geq \theta \geq \tau$. Let n, k, a, c be as follows:

$$n = \frac{2(\theta-\tau)^2((2\theta+1)(\theta-\tau) - 3\theta(\theta+1))}{(\theta-\tau)^2 - \theta^2(\theta+1)^2},$$

$$k = \frac{-\tau((2\theta+1)(\theta-\tau) - \theta(\theta+1))}{(\theta-\tau) + \theta(\theta+1)},$$

$$a = \frac{-\theta(\tau+1)((\theta-\tau) - \theta(\theta+3))}{(\theta-\tau) + \theta(\theta+1)}, \textit{and}$$

$$c = \frac{-(\theta+1)\tau((\theta-\tau) - \theta(\theta+1))}{(\theta-\tau) + \theta(\theta+1)}.$$

If (n, k, a, c) are the parameters of X, then X is said to be a *Smith graph.*

A strongly regular graph is said to be of *pseudo Latin square type* if it has the same parameters as a Latin square graph. A strongly regular graph is said to be of *negative Latin square type* if there exists n and r such that its parameters are

$$(n^2, r(n+1), -n + r^2 + 3r, r^2 + r).$$

4.4.1 Theorem. *Let X be a strongly regular graph and suppose there exists $x \in V(X)$ such that the subconstituents $X_1(x)$ and $X_2(x)$ are both strongly regular. Then, one of the following is true:*

(i) X is isomorphic to the pentagon,

(ii) X is of pseudo or negative Latin square type, or

(iii) X or its complement is a Smith graph.

This does not fully characterize the set of strongly regular graphs with strongly regular subconstituents. The next theorem gives a characterization of graph where $q_{ii}^i = 0$.

4.4.2 Theorem. *Let X be strongly regular graph. Then $q_{ii}^i = 0$ for some $i \in \{1, 2\}$ if and only if X is either a pentagon, a Smith graph or the complement of a Smith graph.*

4.5 Small examples of vanishing Krein parameters

A *generalized quadrangle of order* (s, t), denoted $GQ(s, t)$, is an incidence structure such that

(i) each point is incident with $t + 1$ blocks and any two points are incident with at most one block,

(ii) each block is incident with $s + 1$ points and any two blocks are incident with at most one point, and

(iii) for every $x \in \mathcal{P}$ and L a block in $\mathcal{B}$ not incident with x, there exists a unique pair of point and block, say (y, M), where x is incident with M, M is incident with y and y is incident with L.

The *point graph* of a generalized quadrangle Q is the graph whose vertices are the points of Q and two vertices are adjacent if they are both incident with a block of Q. It is well-known that the point graph of generalized quadrangle of order (s, t) is a strongly regular graph with parameters

$$((s + 1)(st + 1), s(t + 1), s - 1, t + 1)$$

and can be found in [10, p.291]. Some calculations will give that the point graph of a generalized quadrangle of order (s, s^2) or (s^2, s) has $q_{ii}^i = 0$ for some i.

From the table of parameters of strongly regular graphs of [2], we can find all graphs with no more than 1000 vertices such that $q_{ii}^i = 0$ for some i. The information is summarized in Table 4.1. By Theorems 4.0.2 and 4.3.4, these graphs are triply regular.

Since the graphs in Table 4.1 have strongly regular subconstituents, it is natural to look at the parameters of the subconstituents. For generalized quadrangles and most of the other graphs in the table, the first subconstituent is a disjoint union of complete graphs. Hence, it would be more interesting to look at the parameters of the second subconstituent. Given the number

Parameters	Comments	Number of graphs
(16,5,0,2)	Clebsch graph	unique
(27,10,1,5)	Schaefli	unique
(100,22,0,6)	HigmanSims	unique
(112,30,2,10)	$GQ(3,9)$	unique
(162, 56, 10, 24)		unique
(275, 112, 30, 56)	2-graph	unique
(325,68,3,17)	$GQ(4,16)$	at least 1
(640,243,66,108)		no known graphs
(756,130,4,26)	$GQ(5,25)$	2 distinct graphs
(784,116,0,20)		no known graphs
(800,204,28,60)		no known graphs

Table 4.1: All graphs meeting Krein bound with number of vertices less than 1000.

of vertices and the valency, for small graphs, the parameter set of the second subconstituent can be determined and we summarize the parameter classes in Table 4.2

This implies that the 2nd subconstituent of the Schlaefli graph with respect to any vertex is isomorphic to the Clebsch graph and the 2nd subconstituent of the Clebsch graph with respect to any vertex is isomorphic to the Petersen graph.

Parameters of X	Params of $X_2(x)$	Comments
(16,5,0,2)	(10, 3, 0, 1)	unique
(27,10,1,5)	(16, 5, 0, 2)	unique
(100,22,0,6)	(77, 16, 0, 4)	unique
(112,30,2,10)	(81, 20, 1, 6)	unique
(162, 56, 10, 24)	(105, 32, 4, 12)	unique
(275,112,30,56)	(162,56,10,24)	unique
(325,68,3,17)	(256, 51, 2, 12)	at least 1
(640,243,66,108)	(396,135,30,54)	no known graphs
(756,130,4,26)	(625,104,3,20)	at least 1
(784,116,0,20)	(667,96,0,16)	no known graphs
(800,204,28,60)	(595,144,18,40)	no known graphs

Table 4.2: Strongly regular second subconstituents of graphs.

Chapter 5

Open problems and further research

From Theorems 4.0.2 and 4.3.4, we get the following statement about strongly regular triply regular graphs.

5.0.1 Theorem. *Let X be a strongly regular graph. If $q_{ii}^i = 0$ for some $i \in \{1, 2\}$, then X is triply regular.*

While we have looked at many theorems concerning triply regular graphs of diameter 2, there is much less literature concerning triply regular graphs of diameter greater than 2.

We saw that strongly regular graphs with strongly regular subconstituents have been well-studied. The natural generalization is to strongly regular graphs with distance regular subconstituents. It is known that any connected subconstituent of a strongly regular graph has diameter at most 3 and Gardiner et al. characterize this case in [5].

From Table 4.1, we see that there exists two non-isomorphic graph with co-parametric second subconstituents. There are no known exactly of pairs of non-isomorphic graphs with isomorphic second subconstituents. Such examples would be of interest for research on graph isomorphism and graph invariant.

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