Graph theory: helping tourists and map-makers since 1735

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A Taste of π
#epicpiday

The Seven Bridges of Königsberg (1735)

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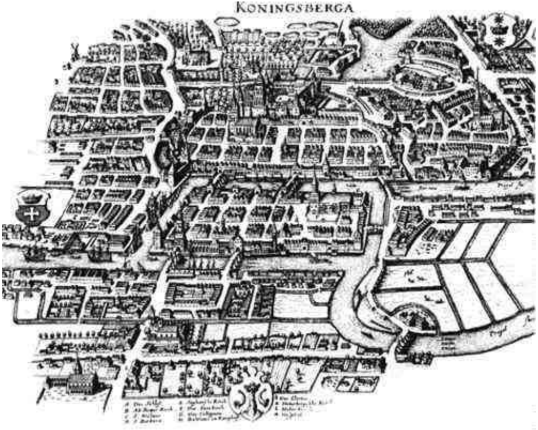

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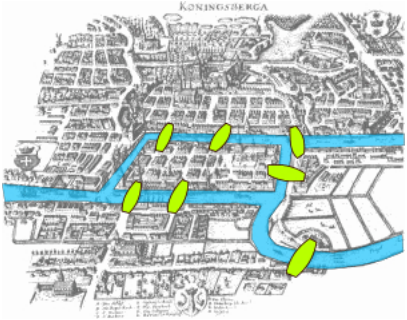

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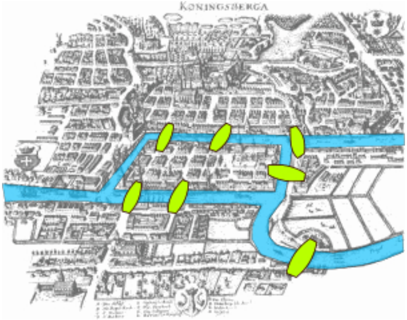
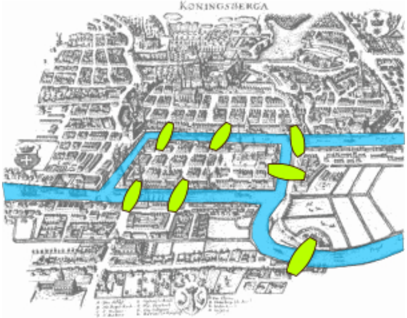
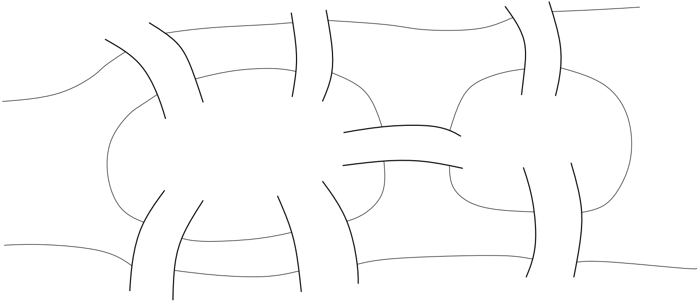
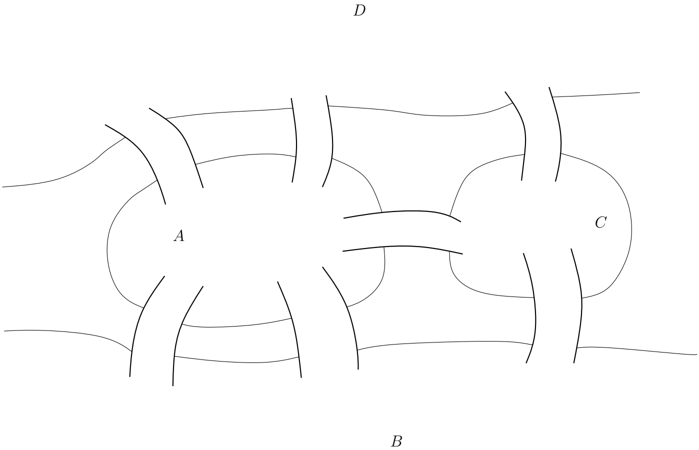
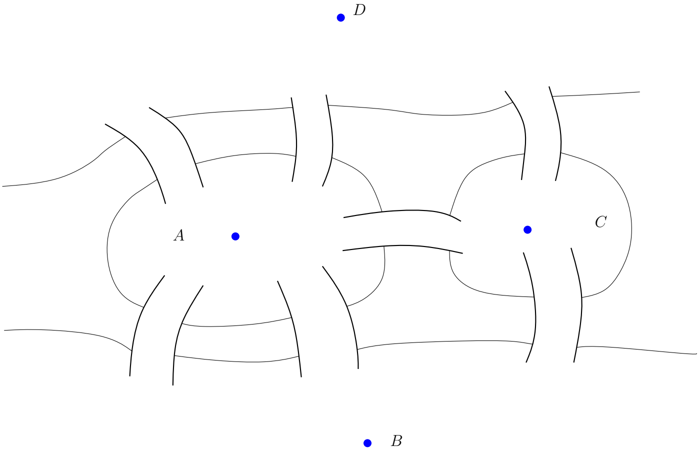


Q: Can you take a walk around Königsberg traversing each bridge exactly once and ending up where you started?






D
A
C
B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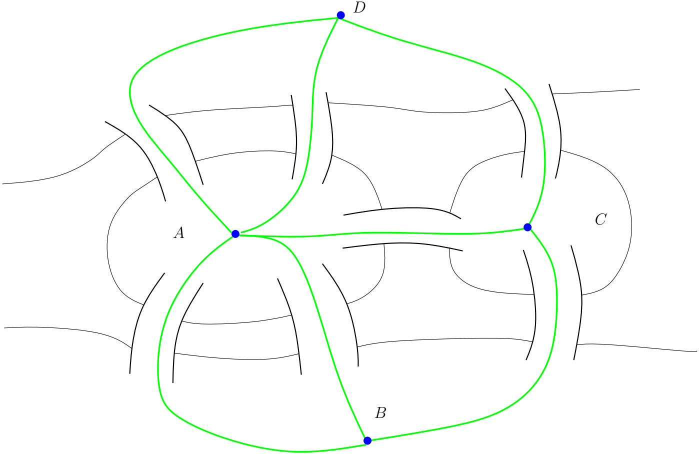

D
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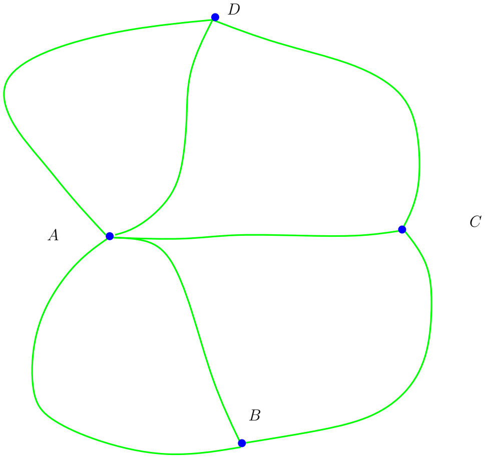
D
A
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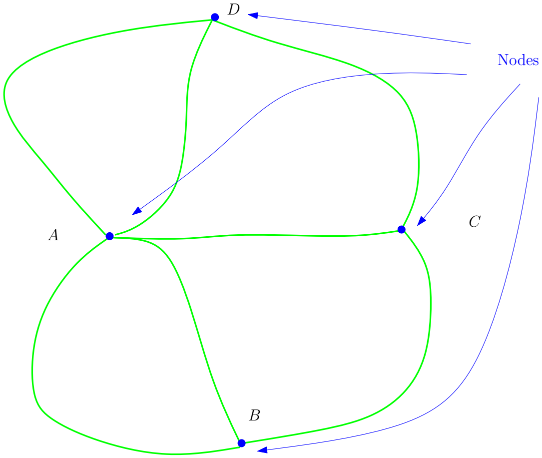
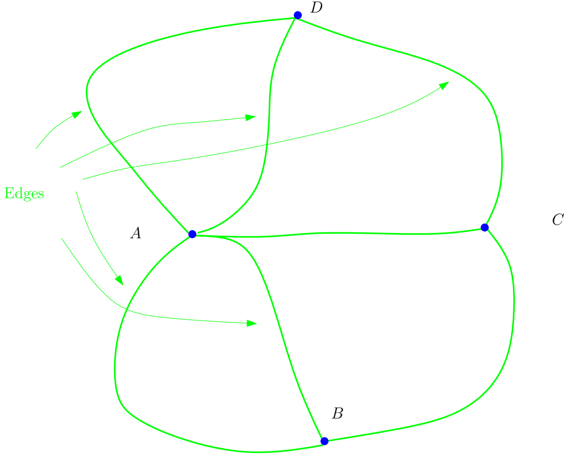
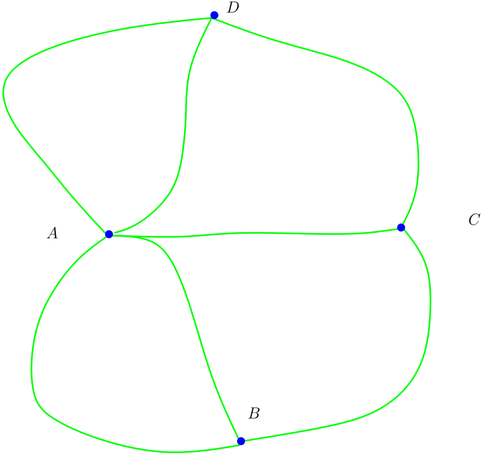

D
Nodes
A
C
B

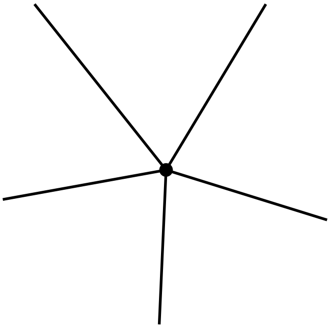


D
Edges
A
C
B

a graph



An eulerian tour is a walk in the graph which traverses every edge exactly once and ends up at the same node that we started at.



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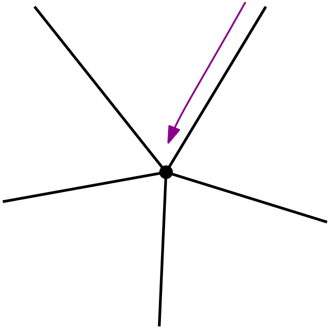

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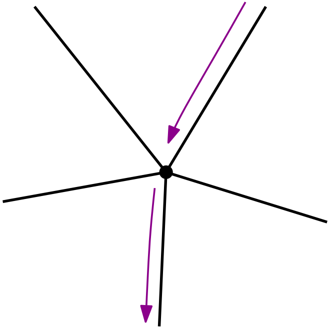

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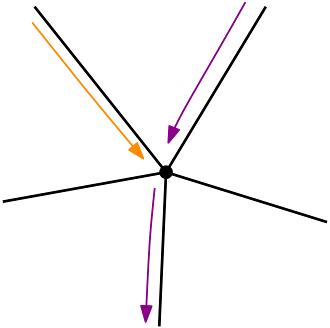

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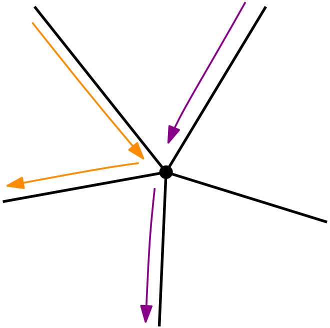

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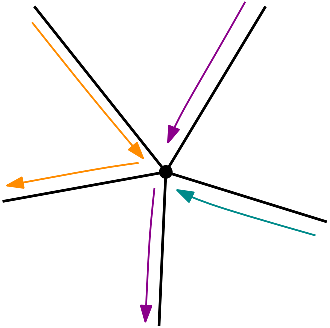

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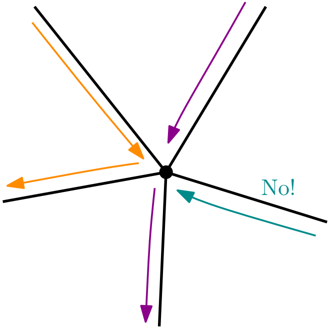

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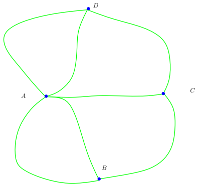

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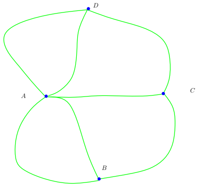
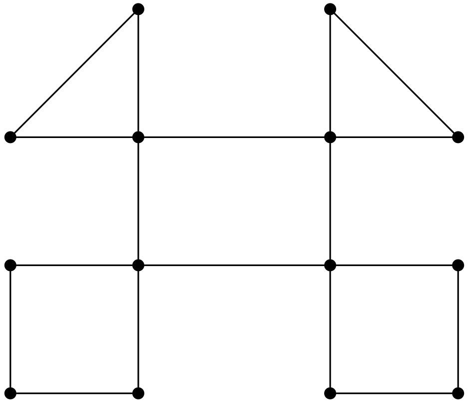


A: — No, there is no eulerian tour through the seven bridges of Königsberg.

Some examples:



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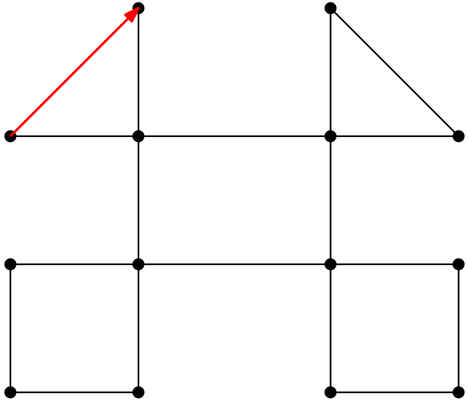

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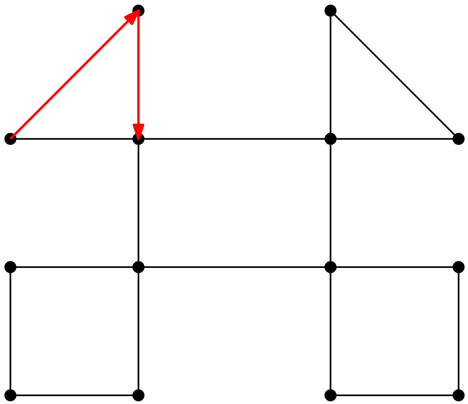

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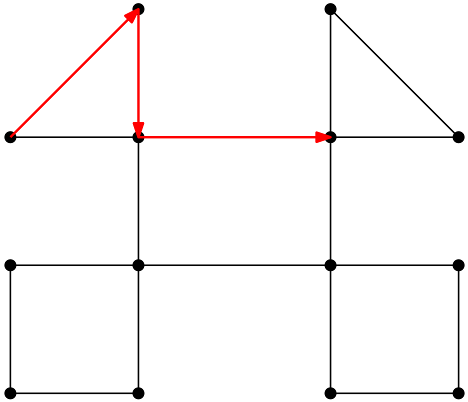

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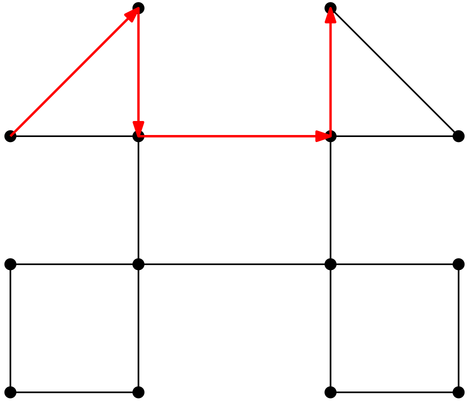

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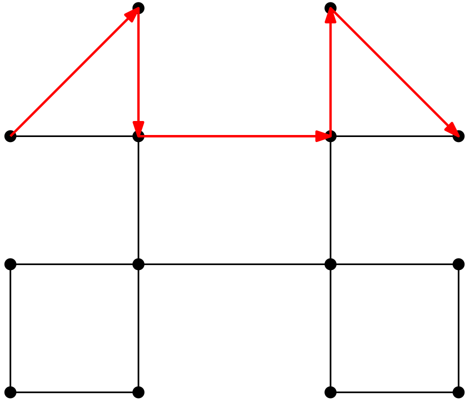

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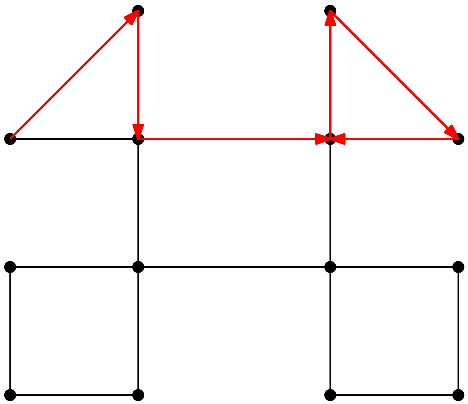

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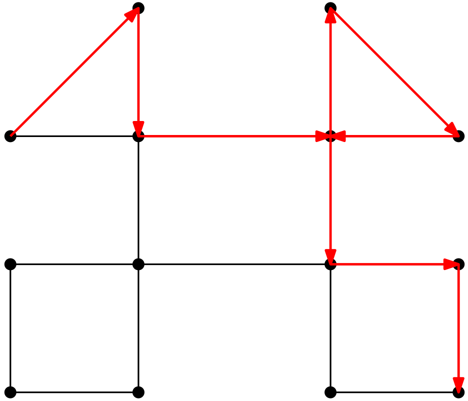

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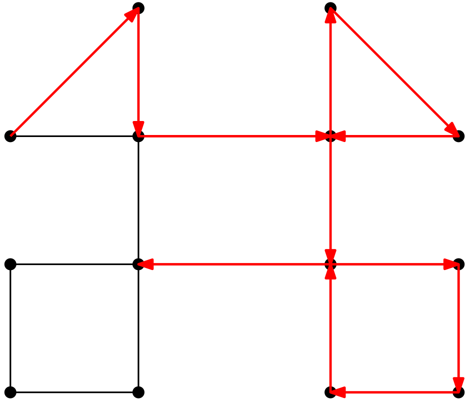

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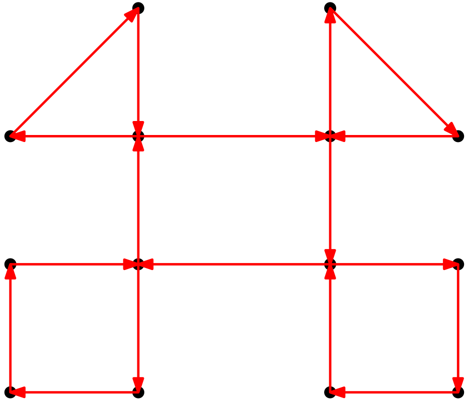

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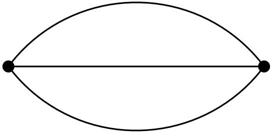

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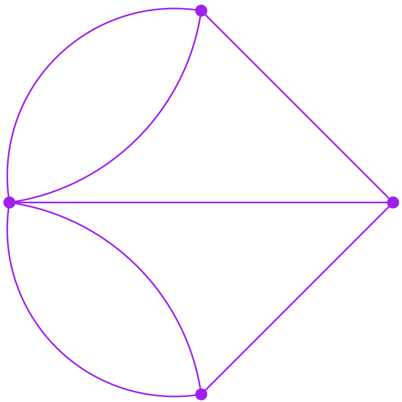

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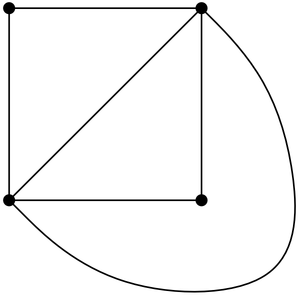

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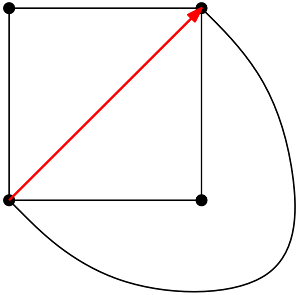

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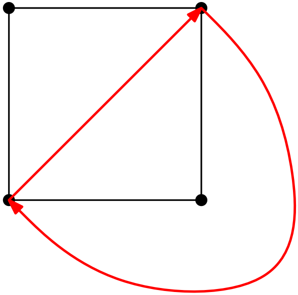

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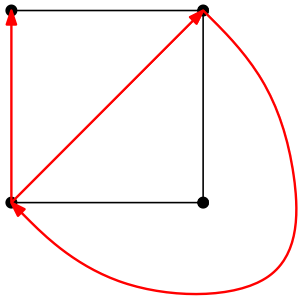

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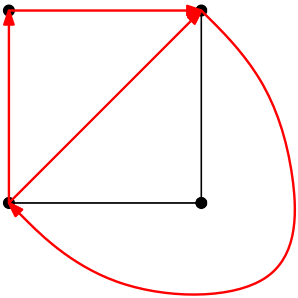

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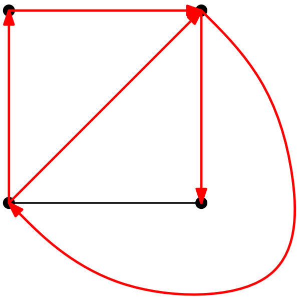

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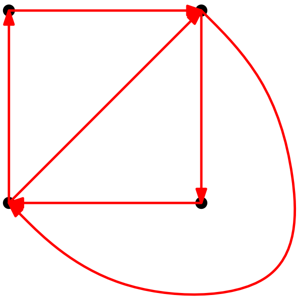

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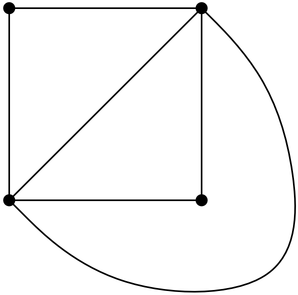

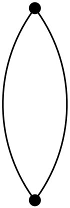

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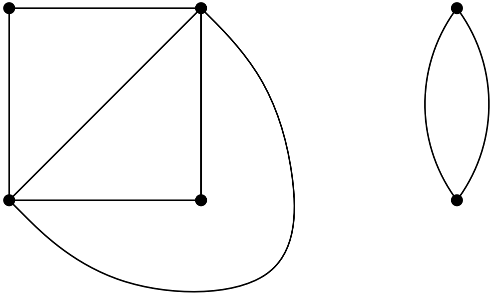

This graph is disconnected and so does not have an eulerian tour.

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Fact	A graph has an eulerian tour if and only if it is connected and every node has even degree.

Why?

Why? Suppose the statement is not true.

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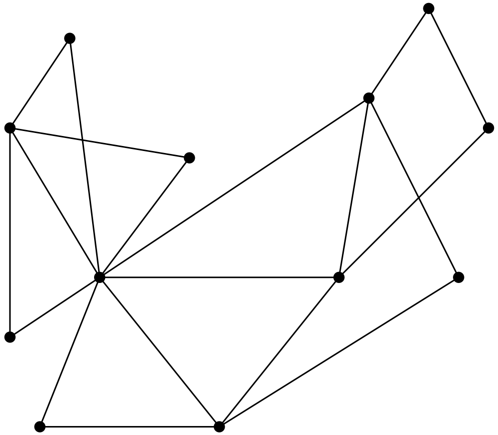

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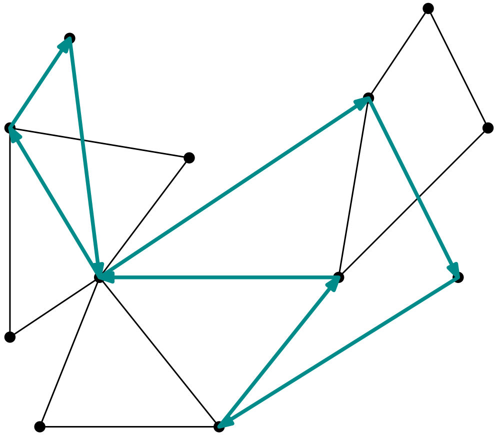

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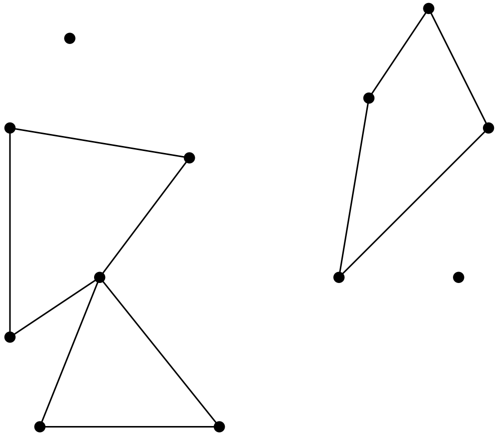

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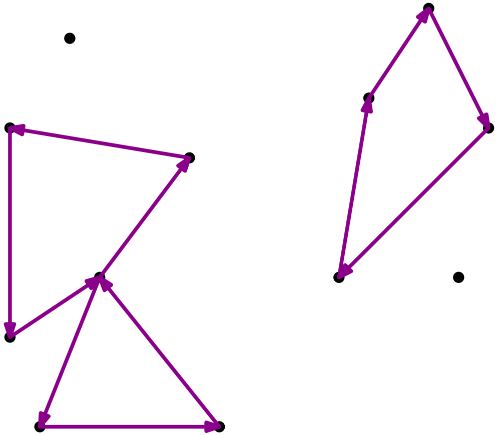

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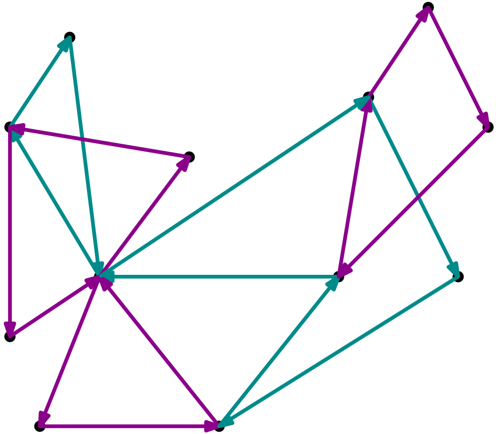

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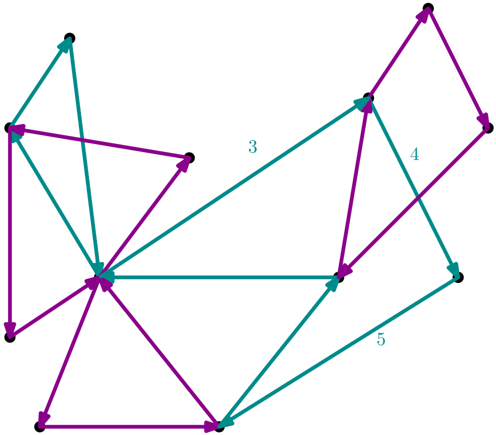

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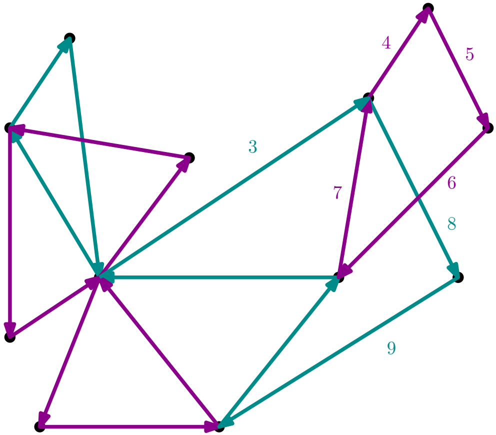

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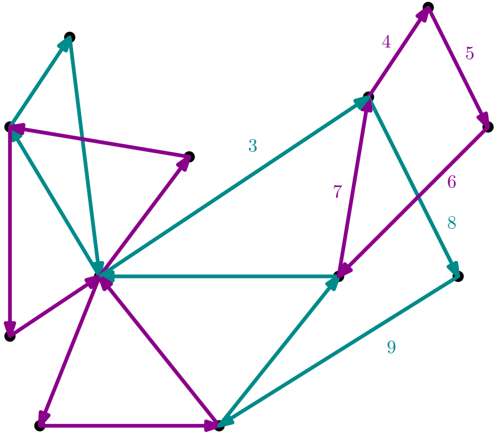
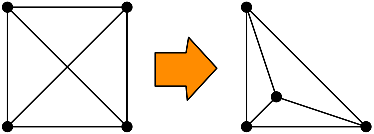


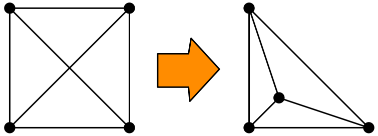
So G must not have been the smallest, a contradiction.

Colouring Planar Eulerian Graphs

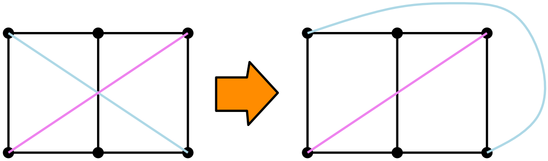
Planar



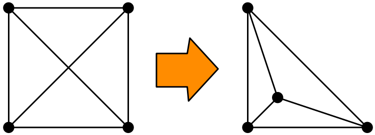
Planar



Non-planar



Planar



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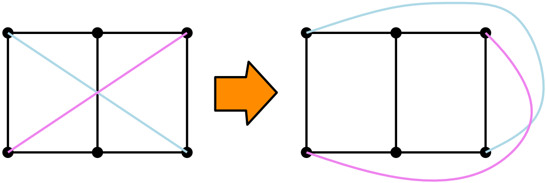

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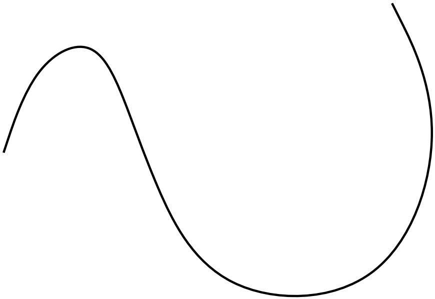

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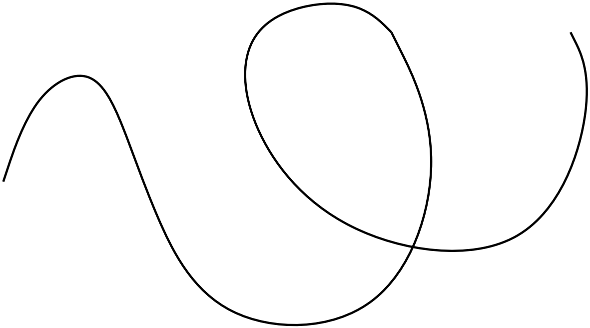

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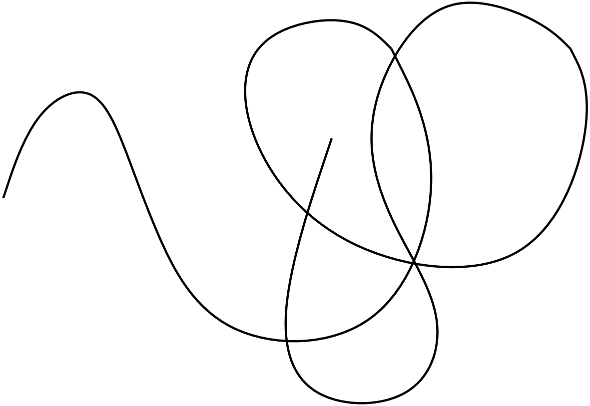

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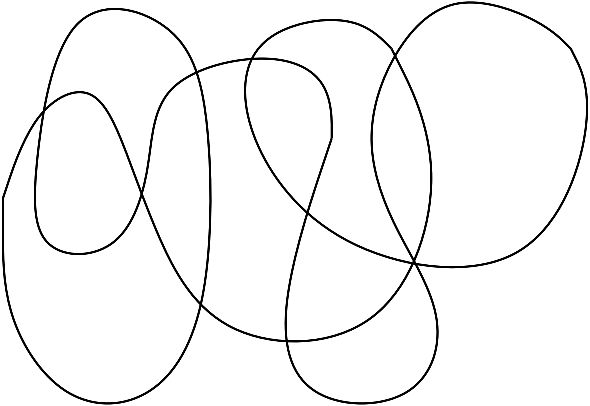

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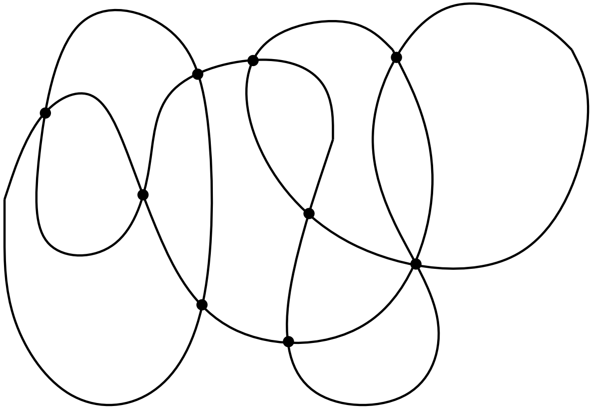

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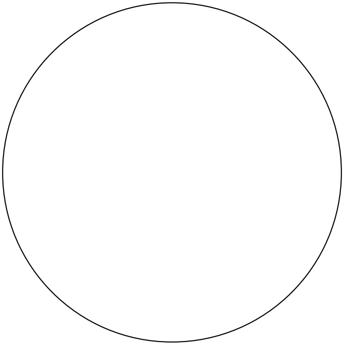

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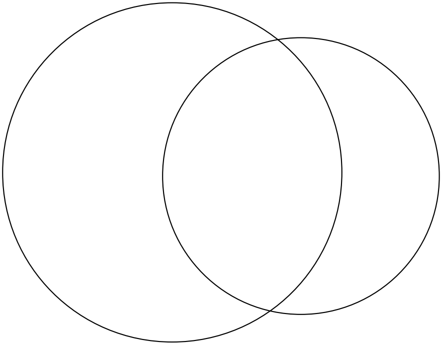

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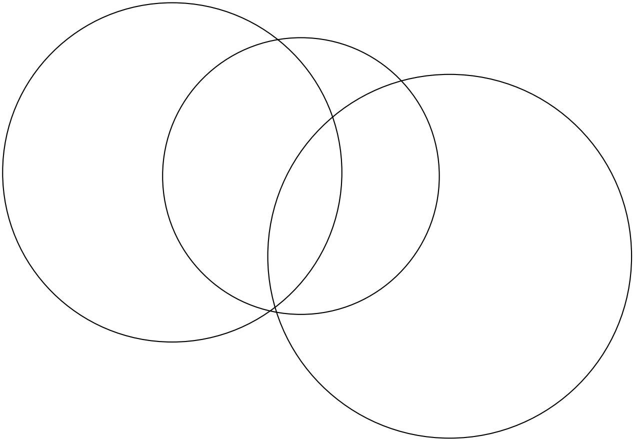

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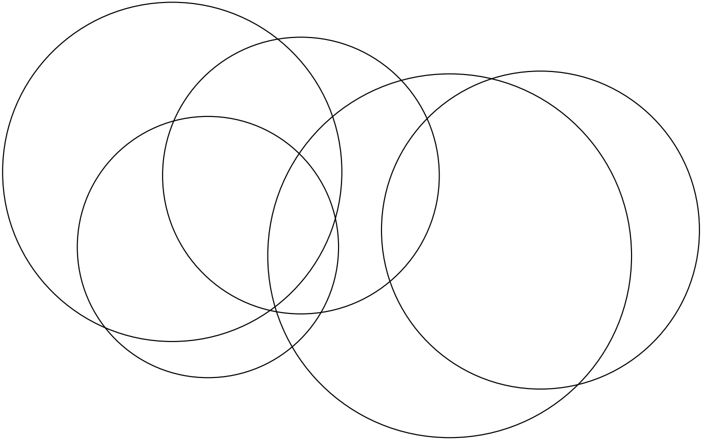

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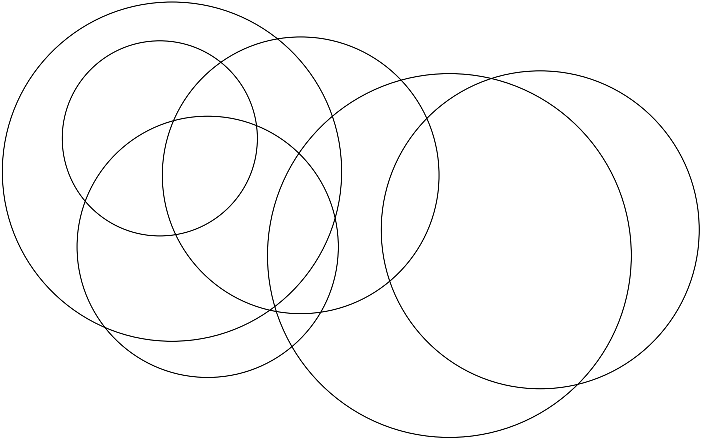

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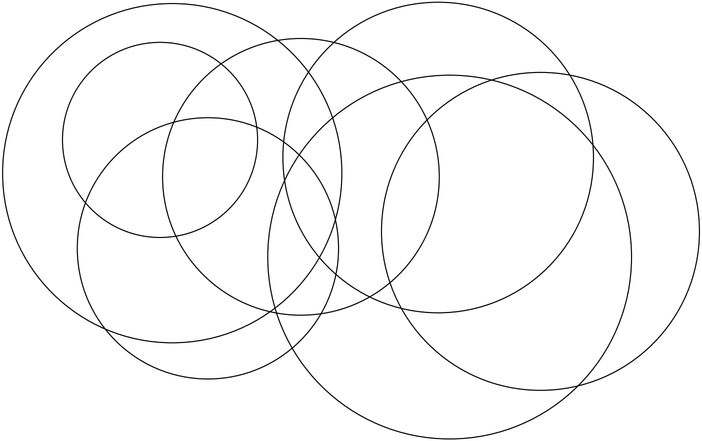

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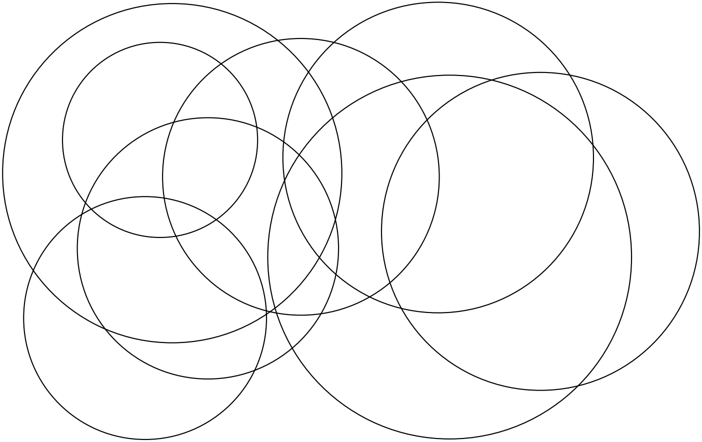

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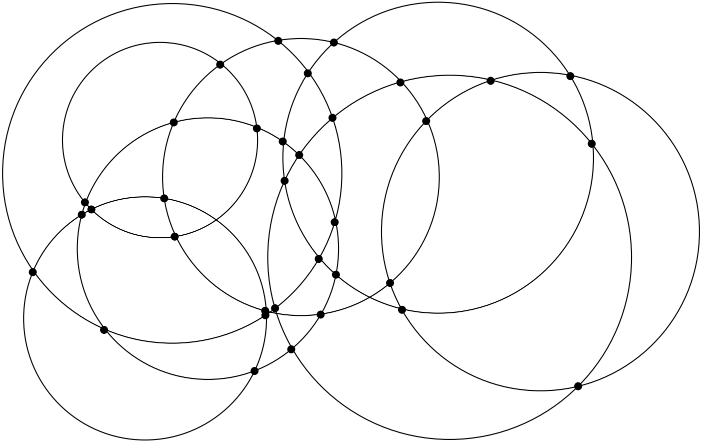

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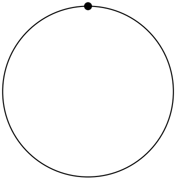

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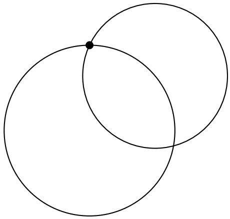

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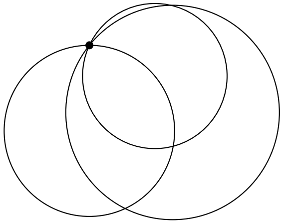

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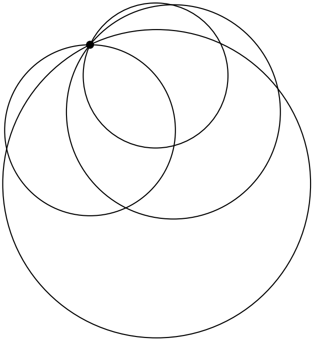

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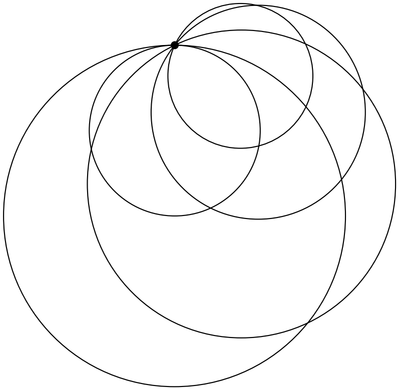

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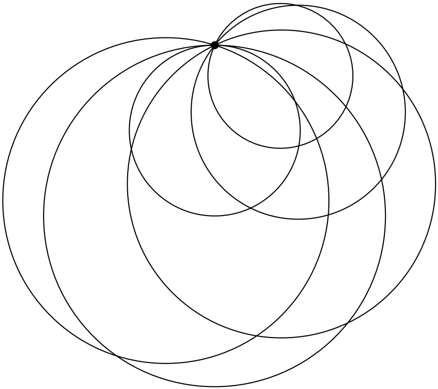

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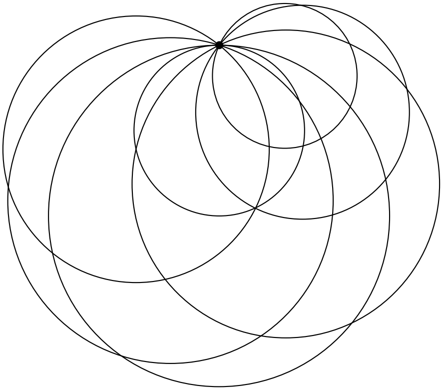

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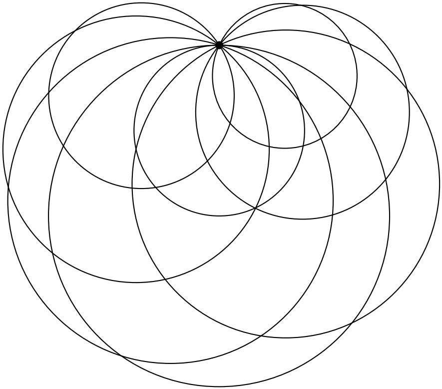

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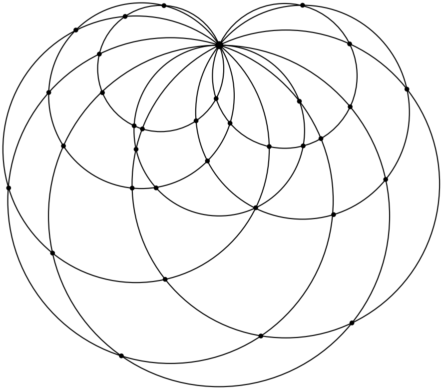
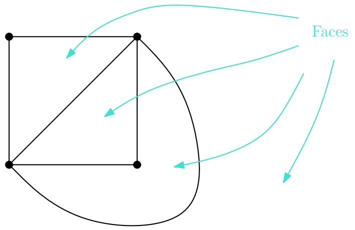
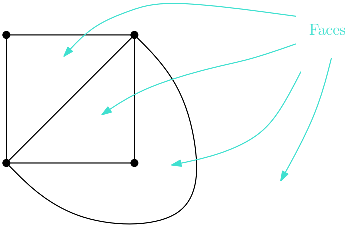
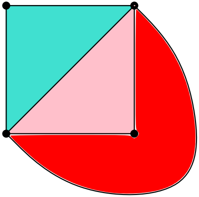



Faces



We want to colour the faces of the graph so that no two faces which share a border have the same colour.



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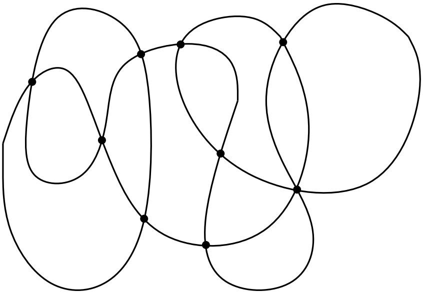
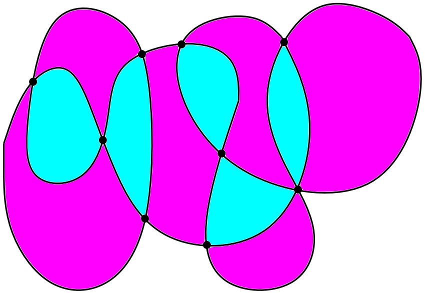
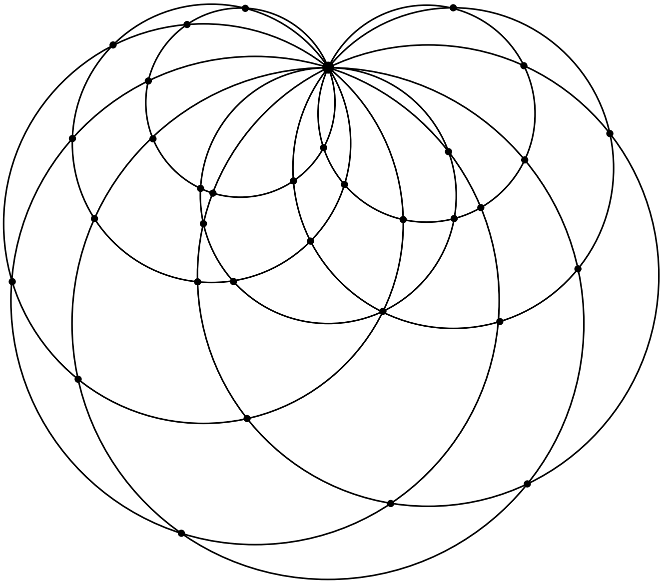
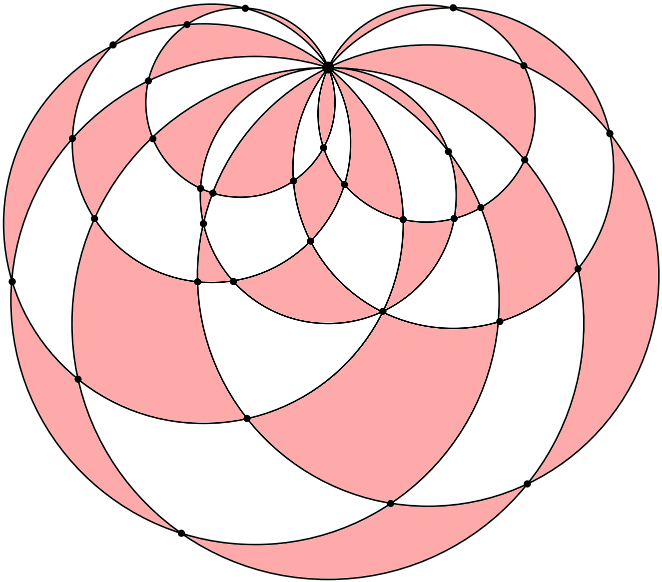
Fact	We can properly colour the faces of a planar eulerian graph with 2 colours.









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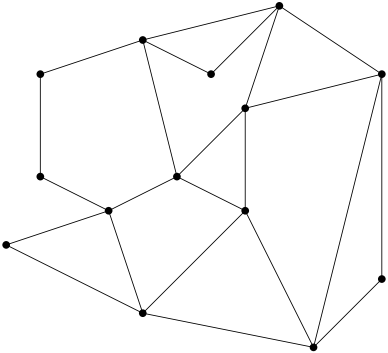

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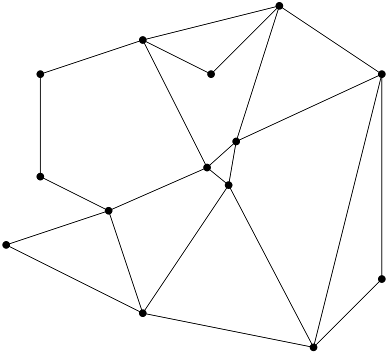

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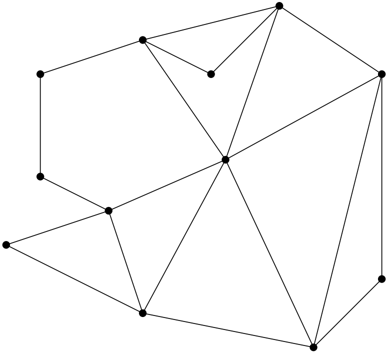

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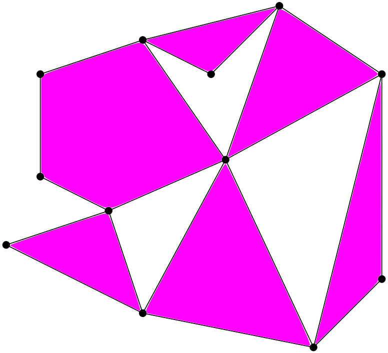

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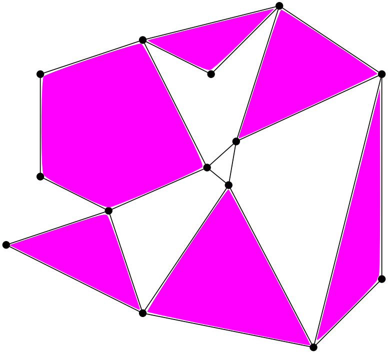

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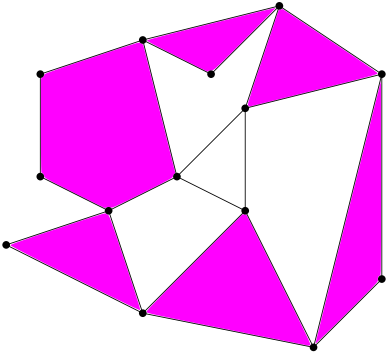

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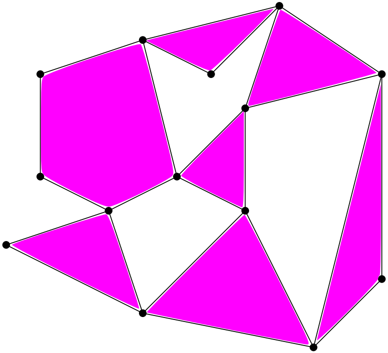

Colouring Planar Graphs

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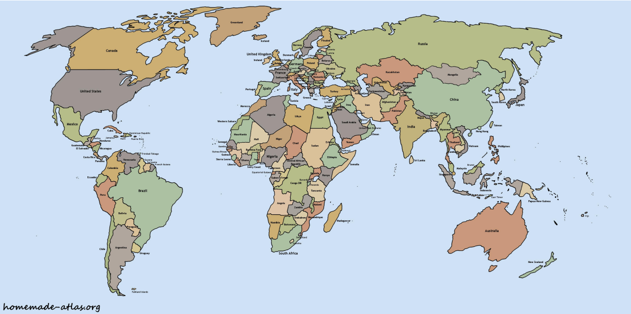

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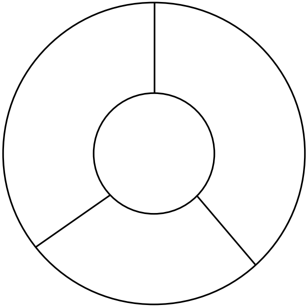

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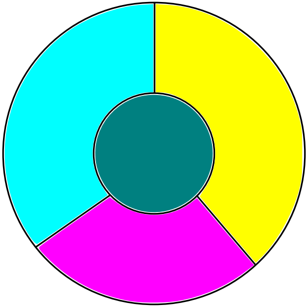

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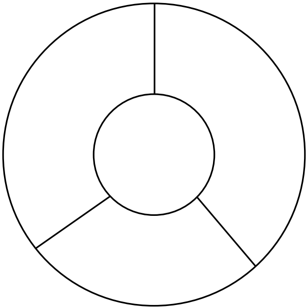

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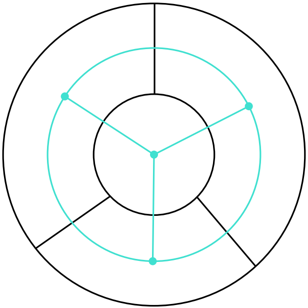

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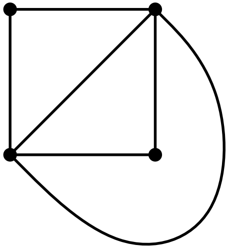

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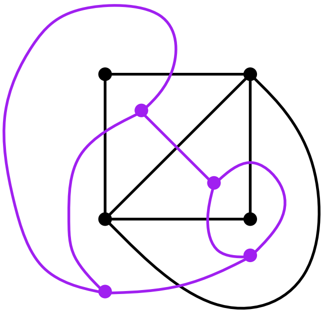

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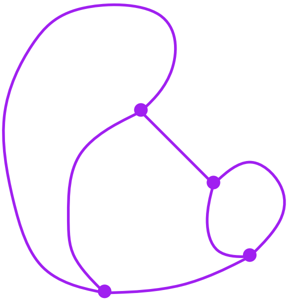

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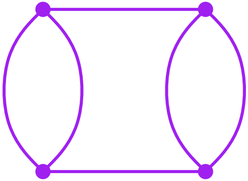

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We will prove the 6 colour theorem today.

Euler’s Theorem

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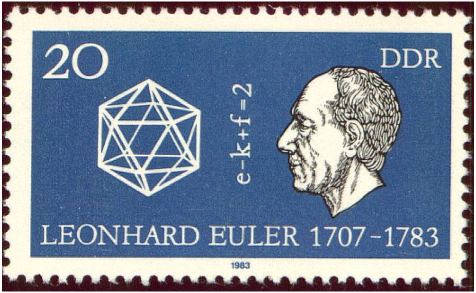

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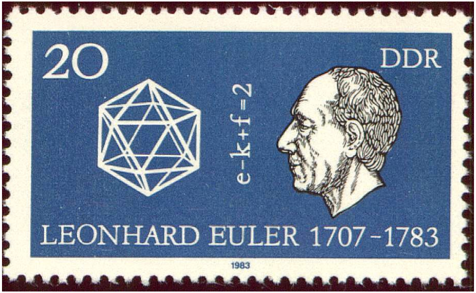

Fact | In a planar graph, # nodes - # edges + # faces $=$ 2.

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From this, we can get that $e \leq 3v - 6$, which implies that there must exist a node of degree at most 5.
Now we may prove the 6 colour theorem.

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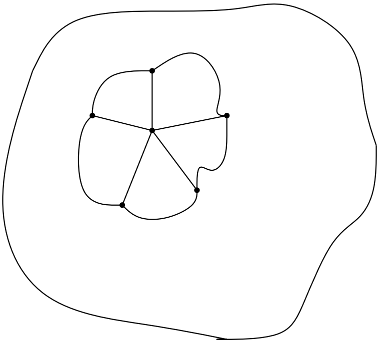

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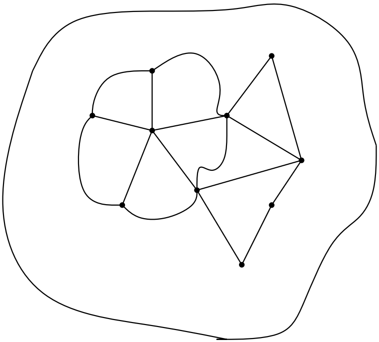

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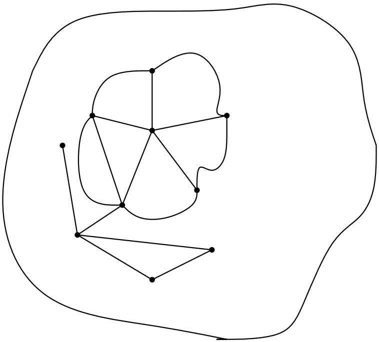

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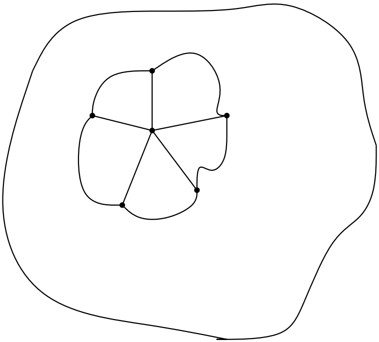

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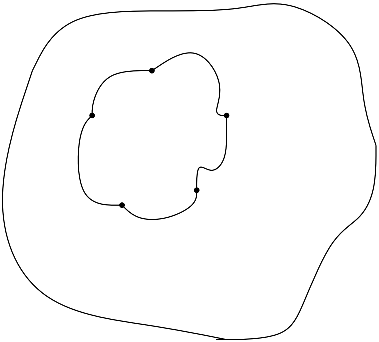

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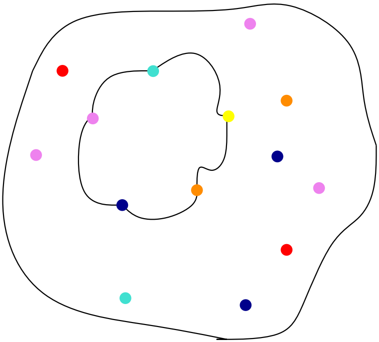

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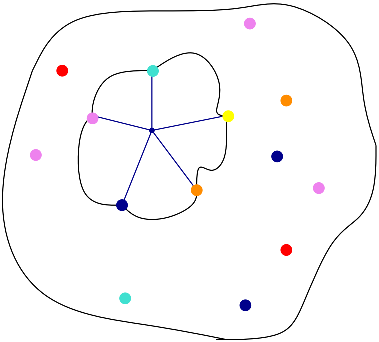

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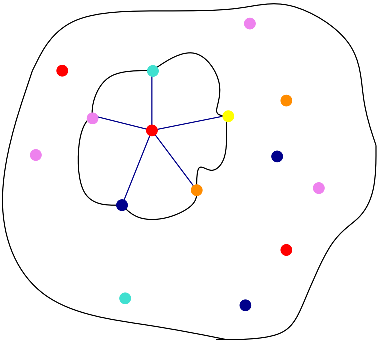

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- Then, we can show, for each of the 633 special configurations, that they cannot occur in the smallest counterexample; if they occurred, we may reduce it and have a smaller counterexample.
- Then, we have a contradiction, and we're done.

Happy epic π day!

Thanks!

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