Graph theory: helping tourists and map-makers since 1735

Krystal Guo

Department of Mathematics
Simon Fraser University
<krystalg@sfu.ca>

A Taste of $\pi$
#epicpiday
The Seven Bridges of Königsberg (1735)
Once upon a time in 1735, in Königsberg, (now Kaliningrad, Russia),
Once upon a time in 1735, in Königsberg, (now Kaliningrad, Russia),
Once upon a time in 1735, in Königsberg, (now Kaliningrad, Russia),
Once upon a time in 1735, in Königsberg, (now Kaliningrad, Russia),
Q: Can you take a walk around Königsberg traversing each bridge exactly once and ending up where you started?
Edges

A
B
C
D
a graph
An eulerian tour is a walk in the graph which traverses every edge exactly once and ends up at the same node that we started at.
An eulerian tour is a walk in the graph which traverses every edge exactly once and ends up at the same node that we started at.
An eulerian tour is a walk in the graph which traverses every edge exactly once and ends up at the same node that we started at.
An **eulerian tour** is a walk in the graph which traverses every edge exactly once and ends up at the same node that we started at.
An eulerian tour is a walk in the graph which traverses every edge exactly once and ends up at the same node that we started at.
An **eulerian tour** is a walk in the graph which traverses every edge exactly once and ends up at the same node that we started at.
An **eulerian tour** is a walk in the graph which traverses every edge exactly once and ends up at the same node that we started at.
The degree of a node is the number of edges at that node.
The **degree** of a node is the number of edges at that node.

| Fact | If a graph has an eulerian tour, then the degree of every node is even. |
The degree of a node is the number of edges at that node.

**Fact**

If a graph has an eulerian tour, then the degree of every node is even.

and thus we have answered our original question:
The **degree** of a node is the number of edges at that node.

### Fact

If a graph has an eulerian tour, then the degree of every node is even.

and thus we have answered our original question:

![Diagram of the seven bridges of Königsberg]

A: No, there is no eulerian tour through the seven bridges of Königsberg.
The **degree** of a node is the number of edges at that node.

### Fact

If a graph has an eulerian tour, then the degree of every node is even.

and thus we have answered our original question:

A: No, there is no eulerian tour through the seven bridges of Königsberg.
Some examples:
Some examples:
Some examples:

This graph is disconnected and so does not have an eulerian tour.
Some examples:
Some examples:
Some examples:
Some examples:
Some examples:
Some examples:
Some examples:
Some examples:
Some examples:
Some examples:
Some examples:
Some examples:
Some examples:
Some examples:
Some examples:
Some examples:
Some examples:
Some examples:

This graph is disconnected and so does not have an eulerian tour.
We’ve see that:

- A graph has an Eulerian tour if and only if it is connected and every node has even degree.
We’ve see that:

- a graph has an **eulerian tour** $\implies$ it has **even degree** at every vertex.
We’ve see that:

- a graph has an **eulerian tour** $\Rightarrow$ it has **even degree** at every vertex.
- a graph has an **eulerian tour** $\Rightarrow$ it is **connected**.

**Fact**

A graph has an eulerian tour if and only if it is connected and every node has even degree.
We’ve see that:

- a graph has an **eulerian tour** $\Rightarrow$ it has **even degree** at every vertex.
- a graph has an **eulerian tour** $\Rightarrow$ it is **connected**.

It turns out that those conditions are sufficient:
We’ve see that:

- a graph has an eulerian tour $\Rightarrow$ it has even degree at every vertex.
- a graph has an eulerian tour $\Rightarrow$ it is connected.

It turns out that those conditions are sufficient:

- a graph has even degree at every vertex and is connected $\Rightarrow$ it has an eulerian tour.
We’ve see that:

- a graph has an **eulerian tour** ⇒ it has **even degree** at every vertex.
- a graph has an **eulerian tour** ⇒ it is **connected**.

It turns out that those conditions are sufficient:

- a graph has **even degree** at every vertex and is **connected** ⇒ it has an **eulerian tour**.

**Fact**

A graph has an eulerian tour if and only if it is connected and every node has even degree.
Why?
Why? Suppose the statement is not true.
Why? Suppose the statement is not true. Then there exists a counterexample.
Why? Suppose the statement is not true. Then there exists a counterexample. So we may consider, $G$, the smallest counterexample; the one with the least number of edges.
Why? Suppose the statement is not true. Then there exists a counterexample. So we may consider, $G$, the smallest counterexample; the one with the least number of edges.
Why? Suppose the statement is not true. Then there exists a counterexample. So we may consider, $G$, the smallest counterexample; the one with the least number of edges.
Why? Suppose the statement is not true. Then there exists a counterexample. So we may consider, $G$, the smallest counterexample; the one with the least number of edges.
Why? Suppose the statement is not true. Then there exists a counterexample. So we may consider, $G$, the smallest counterexample; the one with the least number of edges.
Why? Suppose the statement is not true. Then there exists a counterexample. So we may consider, $G$, the smallest counterexample; the one with the least number of edges.
Why? Suppose the statement is not true. Then there exists a counterexample. So we may consider, $G$, the smallest counterexample; the one with the least number of edges.
Why? Suppose the statement is not true. Then there exists a counterexample. So we may consider, \( G \), the smallest counterexample; the one with the least number of edges.
Why? Suppose the statement is not true. Then there exists a counterexample. So we may consider, $G$, the smallest counterexample; the one with the least number of edges.

So $G$ must not have been the smallest, a contradiction.
Colouring Planar Eulerian Graphs
Planar
Planar

Non-planar
Planar

Non-planar
We consider a planar graph with an eulerian tour.
We consider a planar graph with an eulerian tour. Example:
We consider a planar graph with an eulerian tour.
Example:
We consider a planar graph with an eulerian tour.
Example:
We consider a planar graph with an eulerian tour. Example:
We consider a planar graph with an eulerian tour.
Example:
We consider a planar graph with an eulerian tour.
Example:
We consider a planar graph with an eulerian tour.
Example:
We consider a planar graph with an eulerian tour.
Example:
We consider a planar graph with an eulerian tour.
Example:
We consider a planar graph with an eulerian tour.
Example:
We consider a planar graph with an eulerian tour.

Example:
We consider a planar graph with an eulerian tour. Example:
We consider a planar graph with an eulerian tour. Example:
We consider a planar graph with an eulerian tour.
Example:
We consider a planar graph with an eulerian tour. 
Example:
We consider a planar graph with an eulerian tour.
Example:
We consider a planar graph with an eulerian tour.

Example:
We consider a planar graph with an eulerian tour. Example:
We consider a planar graph with an eulerian tour.
Example:
We consider a planar graph with an eulerian tour. Example:
We consider a planar graph with an eulerian tour.
Example:
We consider a planar graph with an eulerian tour.
Example:
We consider a planar graph with an eulerian tour. Example:
We want to colour the faces of the graph so that no two faces which share a border have the same colour.

Fact

We can properly colour the faces of a planar eulerian graph with 2 colours.
We want to **colour** the faces of the graph so that no two faces which share a border have the same colour.
We want to colour the faces of the graph so that no two faces which share a border have the same colour.

Fact

We can properly colour the faces of a planar eulerian graph with 2 colours.
Proof: Suppose the statement is not true.
Proof: Suppose the statement is not true. Let $G$ be the smallest planar eulerian graph which is not 2-colourable.
Proof: Suppose the statement is not true. Let $G$ be the smallest planar eulerian graph which is not 2-colourable.
Proof: Suppose the statement is not true. Let $G$ be the smallest planar eulerian graph which is not 2-colourable.
Proof: Suppose the statement is not true. Let $G$ be the smallest planar eulerian graph which is not 2-colourable.
Proof: Suppose the statement is not true. Let $G$ be the smallest planar eulerian graph which is not 2-colourable.
Proof: Suppose the statement is not true. Let $G$ be the smallest planar eulerian graph which is not 2-colourable.
Proof: Suppose the statement is not true. Let $G$ be the smallest planar eulerian graph which is not 2-colourable.
Proof: Suppose the statement is not true. Let $G$ be the smallest planar eulerian graph which is not 2-colourable.
Colouring Planar Graphs
Q: What happens if the graph is not eulerian?
What happens if the graph is not eulerian?

This is a problem that has troubled mapmakers since ever.
Q: What happens if the graph is not eulerian?

This is a problem that has troubled mapmakers since ever.
Q: What happens if the graph is not eulerian?

This is a problem that has troubled mapmakers since ever.
Q: What’s the largest number of countries that can all share borders?
Q: What's the largest number of countries that can all share borders?
What’s the largest number of countries that can all share borders?
Instead of colouring faces, we can colour vertices instead by taking a dual.
Instead of colouring faces, we can colour vertices instead by taking a dual.
Instead of colouring faces, we can colour vertices instead by taking a dual.
Instead of colouring faces, we can colour vertices instead by taking a dual.
Instead of colouring faces, we can colour vertices instead by taking a dual.
Instead of colouring faces, we can colour vertices instead by taking a dual.
Instead of colouring faces, we can colour vertices instead by taking a dual.
Fact

Every planar graph can be coloured by 4 colours.

This is the Four Colour Theorem. It was first proposed in 1852. There were many false proofs. Heawood proved the 5 colour theorem in 1890. The Four Colour Theorem was first proven by Appel and Haken in 1976. The proof includes 1936 cases that have to be checked by a computer. Robertson, Sanders, Seymour and Thomas improved this to 633 cases in 1996. It remains one of the most difficult theorems to verify today. We will prove the 6 colour theorem today.
Fact  |  Every planar graph can be coloured by 4 colours.

This is the Four Colour Theorem.
Fact

Every planar graph can be coloured by 4 colours.

This is the Four Colour Theorem. It was first proposed in 1852.
This is the **Four Colour Theorem**. It was first proposed in 1852. There were many false proofs.
Every planar graph can be coloured by 4 colours.

This is the Four Colour Theorem. It was first proposed in 1852. There were many false proofs. Heawood proved the 5 colour theorem in 1890.
Fact | Every planar graph can be coloured by 4 colours.

This is the **Four Colour Theorem**. It was first proposed in 1852. There were many false proofs. Heawood proved the 5 colour theorem in 1890.
This is the **Four Colour Theorem**. It was first proposed in 1852. There were many false proofs. Heawood proved the 5 colour theorem in 1890.

The Four Colour Theorem was first proven by Appel and Haken in 1976. The proof includes 1936 cases that have to be checked by a computer.
This is the Four Colour Theorem. It was first proposed in 1852. There were many false proofs. Heawood proved the 5 colour theorem in 1890.

The Four Colour Theorem was first proven by Appel and Haken in 1976. The proof includes 1936 cases that have to be checked by a computer. Robertson, Sanders, Seymour and Thomas improved this to 633 cases in 1996.
This is the Four Colour Theorem. It was first proposed in 1852. There were many false proofs. Heawood proved the 5 colour theorem in 1890.

The Four Colour Theorem was first proven by Appel and Haken in 1976. The proof includes 1936 cases that have to be checked by a computer. Robertson, Sanders, Seymour and Thomas improved this to 633 cases in 1996. It remains one of the most difficult theorems to verify today.
This is the **Four Colour Theorem**. It was first proposed in 1852. There were many false proofs. Heawood proved the 5 colour theorem in 1890.

The Four Colour Theorem was first proven by Appel and Haken in 1976. The proof includes 1936 cases that have to be checked by a computer. Robertson, Sanders, Seymour and Thomas improved this to 633 cases in 1996. It remains one of the most difficult theorems to verify today.

We will prove the 6 colour theorem today.
Euler’s Theorem
Euler’s Theorem

In a planar graph, $\# \text{nodes} - \# \text{edges} + \# \text{faces} = 2$. 

---

**Euler’s Theorem**

The equation $e - k + f = 2$ relates the number of edges ($e$), vertices ($k$), and faces ($f$) of a planar graph.

---

**Leonhard Euler 1707 - 1783**

The postage stamp commemorates the work of Leonhard Euler, a key figure in mathematics who made significant contributions to graph theory. The stamp features Euler’s famous formula $e - k + f = 2$. 

---

**1983 DDR**

The stamp was issued by the German Democratic Republic (DDR) in 1983 to honor Euler’s 200th anniversary.
Euler’s Theorem

Fact

In a planar graph, \# nodes - \# edges + \# faces = 2.
If $e$ is the number of edges, $n$ is the number of nodes and $f$ is the number of face, then Euler’s theorem says:

$$v - e + f = 2$$
If \( e \) is the number of edges, \( n \) is the number of nodes and \( f \) is the number of face, then Euler's theorem says:

\[
\nu - e + f = 2
\]

From this, we can get that \( e \leq 3\nu - 6 \), which implies that there must exist a node of degree at most 5.
If $e$ is the number of edges, $n$ is the number of nodes and $f$ is the number of face, then Euler’s theorem says:

$$v - e + f = 2$$

From this, we can get that $e \leq 3v - 6$, which implies that there must exist a node of degree at most 5.

Now we may prove the 6 colour theorem.
Fact  Every planar graph can be coloured with 6 colours.

Suppose that it is not true and let $G$ be the smallest counterexample. $G$ has a vertex of degree 5.
Fact

Every planar graph can be coloured with 6 colours.

Suppose that it is not true and let $G$ be the smallest counterexample. $G$ has a vertex of degree 5.
Suppose that it is not true and let $G$ be the smallest counterexample. $G$ has a vertex of degree 5.
Fact

Every planar graph can be coloured with 6 colours.

Suppose that it is not true and let $G$ be the smallest counterexample. $G$ has a vertex of degree 5.
Every planar graph can be coloured with 6 colours.

Suppose that it is not true and let $G$ be the smallest counterexample. $G$ has a vertex of degree 5.
Suppose that it is not true and let $G$ be the smallest counterexample. $G$ has a vertex of degree 5.
Fact | Every planar graph can be coloured with 6 colours.

Suppose that it is not true and let $G$ be the smallest counterexample. $G$ has a vertex of degree 5.
Fact | Every planar graph can be coloured with 6 colours.

Suppose that it is not true and let $G$ be the smallest counterexample. $G$ has a vertex of degree 5.
Suppose that it is not true and let $G$ be the smallest counterexample. $G$ has a vertex of degree 5.
The Four Colour Theorem is very difficult, but it uses a similar idea.
The Four Colour Theorem is very difficult, but it uses a similar idea.
Basic idea:
The Four Colour Theorem is very difficult, but it uses a similar idea.

Basic idea:

- We also suppose the statement is untrue and consider the smallest counterexample.
The Four Colour Theorem is very difficult, but it uses a similar idea.

Basic idea:

- We also suppose the statement is untrue and consider the smallest counterexample.
- It’s possible to show that any planar graph that cannot be coloured with 4 colours must contain one of 633 special configurations.
The Four Colour Theorem is very difficult, but it uses a similar idea.

Basic idea:

- We also suppose the statement is untrue and consider the smallest counterexample.
- It’s possible to show that any planar graph that cannot be coloured with 4 colours must contain one of 633 special configurations.
- Then, we can show, for each of the 633 special configurations, that they cannot occur in the smallest counterexample; if they occurred, we may reduce it and have a smaller counterexample.
The Four Colour Theorem is very difficult, but it uses a similar idea.

Basic idea:

▶ We also suppose the statement is untrue and consider the smallest counterexample.
▶ It’s possible to show that any planar graph that cannot be coloured with 4 colours must contain one of 633 special configurations.
▶ Then, we can show, for each of the 633 special configurations, that they cannot occur in the smallest counterexample; if they occurred, we may reduce it and have a smaller counterexample.
▶ Then, we have a contradiction, and we’re done.
Happy epic $\pi$ day!
Thanks!

Some pictures from Wikimedia Commons.