KRYSTAL GUO

COMBINATORIAL ENUMERATION

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Note to my students

The lecture notes from class will be placed here, after a bit of formatting to make it nicer to read.

1 The Basic of Counting

1.1 Binomial coefficients

We want to count things. In general, we describe some class of objects, which have an order, then we count all of them of a given order.

Examples:

1. All binary strings of length n.
2. All binary strings of length n, which do not contain two consecutive 0s.
3. k-subsets of an n element set.

1.1.1 Theorem. *For non-negative integers n and k, the number of k-element subsets of an n-element set is*

$$\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \frac{n!}{(n-k)!k!}.$$

We denote this ratio by $\binom{n}{k}$, which we read as "n choose k".

Proof. Let $\mathcal{L}$ be the set of all ordered lists of k distinct elements from $\{1,\dots,n\}$. For the first element in the list, we have n choices. For the second one, we have $n-1$ choices, and $n-2$ for the third element, etc. In general, when choosing the ith number for the list, we have already used $i-1$ numbers and so we have $n-i+1$ numbers remaining to choose from. Thus

$$|\mathcal{L}| = n(n-1)\cdots(n-k+1).$$

We can count the elements of $\mathcal{L}$ in another way. We can choose k numbers that will appear in the list, and then we order the number in all possible ways. Suppose there are N ways to choose k elements from $\{1,\dots,n\}$. There are $k!$ way to order, or permute, each k-element set. Thus

$$|\mathcal{L}| = N(k!).$$

We have obtained from these two expressions for $|\mathcal{L}|$ that

$$N = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

as required. □

Note that $\binom{n}{k} = 0$ whenever $k > n$. The empty sum is 0 and the empty product is 1;

$$\sum_{x \in \emptyset} x = 0, \quad \prod_{x \in \emptyset} x = 1.$$

Thus, $x^0 = 1$ and $0! = 1$. Also, $\binom{n}{0} = 1$. For $0 \leq k \leq n$, we have

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}.$$

This makes intuitive sense; the number of ways to select k elements is equal to the number of ways to discard $n-k$ elements.

The proof of Theorem 1.1.1 illustrates how many problems in enumeration are solved

- We want to count something, but it's hard.
- We can count an easier set of objects (lists).
- We relate the two quantities.

Unfortunately, these reductions are ad-hoc and not easy to find. What to do? What if we want a way that solves many problems? That is the goal of this course.

We make a calculus for enumeration problems; we convert enumeration problems to elementary algebra. Let's start with an easy example.

There are 8 subsets of $\{1,2,3\}$; namely:

$$\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}.$$

Now let $f(y_1, y_2, y_3) = (1+y_1)(1+y_2)(1+y_3)$ and we expand

$$f(y_1, y_2, y_3) = 1 + y_1 + y_2 + y_3 + y_1y_2 + y_1y_3 + y_2y_3 + y_1y_2y_3.$$

There is a natural one-to-one correspondence between the subsets of $\{1,2,3\}$ and the terms in the expansion of $f(y_1, y_2, y_3)$. We will take advantage of it to count things. For example,

$$f(1,1,1) = (1+1)^3 = 8$$

and thus there are 8 subsets of $\{1,2,3\}$. Next, we have

$$f(x,x,x) = (1+x)^3 = 1 + 3x + 3x^2 + x^3.$$

The coefficient of x^k counts the number of k-elements subsets of $\{1,2,3\}$.

1.1.2 Theorem. *For any non-negative integer n,*

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Proof. Let $y_1, \ldots, y_n$ be variables and, for any subset S of $\{1, \ldots, n\}$, let

$$y_S = \prod_{i \in S} y_i.$$

So, $y_{\{1,2,3\}} = y_1 y_2 y_3$ and $y_\emptyset = 1$. From this correspondence, we get

$$(1+y_1)(1+y_2)\cdots(1+y_n) = \sum_{S\subseteq\{1,\ldots,n\}} y_S.$$

We substitute $y_i = x$ for all i; then y_S will become $x^{|S|}$. Hence,

$$(1+x)^n = \sum_{S\subseteq\{1,\ldots,n\}} x^{|S|}.$$

Using Theorem 1.1.1, we see that

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

as claimed. □

We often write $[n]$ for $\{1,2,\ldots,n\}$.

This proof was a "combinatorial proof"; we do some counting and we prove an algebraic identity. Let's do it again, but a bit more complicated.

1.1.3 Theorem. *For non-negative integers n, k,*

$$\binom{n+k}{n} = \sum_{i=0}^{k} \binom{n+i-1}{n-1}.$$

Proof. Let $\mathcal{S}$ denote the set of all n-element subsets of $[n+k]$. By Theorem 1.1.1, we have that

$$|\mathcal{S}| = \binom{n+k}{n}.$$

Consider a n-subsets $A \in \mathcal{S}$. The largest number in A is at least n. For each $i \in \{0,\ldots,k\}$, let $\mathcal{S}_i$ be the. set of all sets in $\mathcal{S}$ where the largest element is $n+i$. Thus $(\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_k)$ is a partition of $\mathcal{S}$ and thus

$$|\mathcal{S}| = |\mathcal{S}_0| + |\mathcal{S}_1| + \cdots + |\mathcal{S}_k|.$$

Now if $A \in \mathcal{S}_i$, then $n+i \in A$ and $A \setminus \{n+i\}$ is a $(n-1)$-element subset of $\{1,2,\ldots,n+i-1\}$. Then Theorem 1.1.1 gives us that

$$|\mathcal{S}_i| = \binom{n+i-1}{n-1}.$$

Hence,

$$\begin{aligned} |\mathcal{S}| &= |\mathcal{S}_0| + |\mathcal{S}_1| + \cdots + |\mathcal{S}_k| \\ &= \binom{n-1}{n-1} + \binom{n}{n-1} + \cdots + \binom{n+k-1}{n-1} \end{aligned}$$

and the result follows. □

Let $A(x) = \sum_{n\geq 0} a_n x^n$ be a series (or a polynomial), then we write

$$[x^n]A(x) = a_n.$$

The coefficients of polynomials and series have meaning for us and this is how we extract them.

1.2 Generating function

Here's the general enumeration problem we want to solve:
Problem: Suppose that S is a set of "configurations" and for each configuration $\sigma \in S$, we have a non-negative integer weight $w(\sigma)$. For a given integer k, how many elements of S have weight k.

Later in the course, we will say more on what is a combinatorial class. If this interests you, you would enjoy category theory. For now, just S to be some family of objects, so that you can assign weights to each element.

1.2.1 Example. Let S be all subsets of $[n]$. For $\sigma \in S$, its weight $w(\sigma) = |\sigma|$. We found that the number of elements of S of weight k is $\binom{n}{k}$.

Let S be a set of configurations with a weight function w. The *generating function* of S with respect to w is defined as follows:

$$\Phi_S(x) = \sum_{\sigma \in S} x^{w(\sigma)}.$$

We can collect like terms (powers of x) in $\Phi_S(x)$ and obtain

$$\Phi_S(x) = \sum_{\sigma \in S} x^{w(\sigma)} = \sum_{k \geq 0} \left(\sum_{\sigma \in S, w(\sigma) = k} 1 \right) x^k = \sum_{k \geq 0} a_k x^k$$

where a_k is the number of configuration in S with weight k.

Let S be all subsets of $[n]$. This is sometimes known as the *power set*. Let the weight of $A \in S$ be $w(A) = |A|$. Then

$$\Phi_S(x) = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$$

Thus,

$$\Phi_S(x) = \sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n.$$

Let S be the set of all integer pairs (a, b) where $a \in \{1, 3, 5\}$ and $b \in \{2, 4, 6\}$. Let $w((a, b)) = a + b$. Let's write the generating function of S with respect to w.

$$\Phi_S(x) = x^{1+2} + x^{1+4} + x^{1+6} + x^{3+2} + x^{3+4} + x^{3+6} + x^{5+2} + x^{5+4} + x^{5+6}.$$

Collecting like terms we get,

$$\Phi_S(x) = x^3 + 2x^5 + 3x^7 + 2x^9 + x^{11}$$

which you can see factors as follows:

$$\Phi_S(x) = (x^1 + x^3 + x^5)(x^2 + x^4 + x^6).$$

Why is this the case and can we use it?

1.2.2 Theorem. *Let $\Phi_S(x)$ be the generating function for a finite set S with respect to a weight function w. Then*

(a) $\Phi_S(1) = |S|$;

(b) *the sum of the weights of all the elements of S is equal to $\Phi'_S(1)$; and*

(c) *the average weight of an element in S is $\frac{\Phi'_S(1)}{\Phi_S(1)}$.*

Proof. Note that

$$\Phi_S(1) = \sum_{\sigma \in S} 1^{w(\sigma)} = |S|.$$

Similarly,

$$\Phi'_S(x) = \sum_{\sigma \in S} w(\sigma) s^{w(\sigma)-1}.$$

Thus,

$$\Phi'_S(1) = \sum_{\sigma \in S} w(\sigma)$$

as required. Part (c) follows from (a) and (b). □

What can we get out of this? Let's return to the example where S is the power set of $\{1, \ldots, n\}$ and $w(\sigma) = |\sigma|$ for $\sigma \in S$. Since the weight is the cardinality, there are $\binom{n}{k}$ elements of weight k and we can write

$$\Phi_S(x) = \sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n.$$

Now we see that $|S| = \Phi_S(1) = 2^n$. Further,

$$\Phi'_S(x) = n(1+x)^{n-1},$$

and so

$$\Phi'_S(1) = n(2)^{n-1},$$

and so the average weight of an element of S is

$$\frac{\Phi'_S(1)}{\Phi_S(1)} = \frac{n(2)^{n-1}}{2^n} = \frac{n}{2}.$$

1.3 *Formal Power Series*

We've said we will make a calculus to do enumeration. Here it is.

When S is a finite set, $\Phi_S(x)$ is a polynomial. When S is infinite, we need to deal with some subtleties. We assume that there are only finitely many elements of S of any given weight, so that the generating is well-defined.

Let $(a_0, a_1, a_2, \ldots)$ be a sequence of rational numbers. The expression

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

is said to be a *formal power series*. We say that a_n is the *coefficient* of x^n in $A(x)$ and we write $a_n = [x^n]A(x)$. The formal power seires is just another way to write a sequence of rational numbers, usually for the purposes of manipulation.

Two formal series are equal iff they have the same sequence of coefficients.

Formal power series are trying to be polynomials. If $A(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ and $B(x) = b_0 + b_1 x + b_2 x^2 + \cdots$ are formal power series, then we define

$$A(x) + B(x) = \sum_{n \geq 0} (a_n + b_n) x^n, \qquad (1.3.1)$$

and

$$A(x)B(x) = \sum_{n\geq 0}\left(\sum_{k\geq 0}^{n} a_k b_{n-k}\right) x^n. \tag{1.3.2}$$

Please note that these are definition. Luckily, $A(x)$ and $B(x)$ are polynomials, these definitions agree with their usual usage. Addition is clear; for multiplication, we see

$$\begin{aligned} A(x)B(x) &= \left(\sum_{i\geq 0} a_i x^i\right)\left(\sum_{j\geq 0} b_j x^j\right) \\ &= \sum_{i\geq 0}\sum_{j\geq 0} a_i b_j x^{i+j} \\ &= \sum_{n\geq 0}\sum_{k\geq 0} a_k b_{n-k} x^n \\ &= \sum_{n\geq 0}\left(\sum_{k\geq 0} a_k b_{n-k}\right) x^n. \end{aligned}$$

We needed some care in defining infinite sums and products, but here there is no ambiguity; there are only finitely many terms contributing to any coefficient in the resulting formal power series. Thus $A(x) + B(x)$ and $A(x)B(x)$ are well-defined.

What other operations do polynomials have? We also want to compose $A(x)$ and $B(x)$ as follows:

$$A(B(x)) = a_0 + a_1 B(x) + a_2 (B(x))^2 + \cdots.$$

However, unlike the case for polynomials, this is not always well-defined. Take $A(x) = 1 + x + x^2 + \cdots$ and let $B(x) = 1 + x$. Then,

$$A(B(x)) = 1 + (1 + x) + (1 + x)^2 + \cdots.$$

What is the coefficient of x^0 here? There are non-zero contributions from an infinite number of terms, so $A(B(x))$ is not a formal power series. The solution to this issue is we forbid all the bad cases. The problem here was that $B(x)$ has a non-zero constant term.

1.3.1 Theorem. *If $A(x)$ and $B(x)$ are formal power series and the constant term of $B(x)$ equals 0, then $A(B(x))$ is a formal power series.*

Proof. It suffices to show that each of the coefficients have non-zero contributions from finitely many terms. We can write $A(x) = a_0 + a_1 x + a_2 x^2 + \cdot$ and $B(x) = xC(x)$, for some formal power series $C(x)$. Then

$$A(B(x)) = a_0 + a_1 xC(x) + a_2 x^2 (C(x))^2 + \cdots.$$

For each $k \geq 0$, note $a_k x^k (C(x))^k$ is a formal power series (since it is the product of $k + 1$ formal power series). Moreover, for each $n < k$, we have

$$[x^n] a_k x^k (C(x))^k = 0.$$

Thus,

$$\begin{aligned} [x^n]A(B(x)) &= [x^n](a_0 + a_1 xC(x) + a_2 x^2 (C(x))^2 + \cdots) \\ &= [x^n](a_0 + a_1 xC(x) + a_2 x^2 (C(x))^2 + \cdots + a_n x^n (C(x))^n) \end{aligned}$$

We see that this expression for the coefficient of x^n is a formal power series because it is a finite sum of formal power series and so it is a finite number. □

Polynomials also have division. This is slightly more complicated. We start by taking inverses.

We say that $B(x)$ is the inversion of $A(x)$ if $A(x)B(x) = 1$; we write $B(x) = A(x)^{-1}$ or $B(x) = \frac{1}{A(x)}$.

1.3.2 Example. Show that the inverse of $1 - x$ is

$$1 + x + x^2 + \cdots .$$

Solution. We observe

$$(1-x)(1+x+x^2+\cdots) = (1+x+x^2+\cdots) - x(1+x+x^2+\cdots) = 1,$$

as required. □

The formal power series $1 + x + x^2 + \cdots$ is the generating function for the set of non-negative integers. We will also frequently consider $S = \{0, \ldots, k\}$, whose generating is

$$\Phi_S(x) = 1 + x + x^2 + \cdots + x^k.$$

We observe that

$$\begin{aligned} \frac{1-x^{k+1}}{1-x} &= (1-x^{k+1})(1+x+x^2+\cdots) \\ &= (1+x+x^2+\cdots) - x^{k+1}(1+x+x^2+\cdots) \\ &= 1+x+x^2+\cdots+x^k. \end{aligned}$$

We may refer to this as the (infinite) Geometric Series. We will now use $(1-x)^{-1}$ to compute other inverses.

1.3.3 Example. Determine the inverse of $1 - x + 2x^2$.

Solution. Let $P(x) = 1 - x$. Note that $1 - x + 2x^2 = 1 - (x - 2x^2) = P(x - 2x^2)$. Using that $P(x)^{-1} = 1 + x + x^2 + \cdots$, we have that

$$(1 - x + 2x^2)^{-1} = P(x - 2x^2)^{-1} = 1 + (x - 2x^2) + (x - 2x^2)^2 + \cdots$$

We could have simplified more, but we only wish to demonstrate how to compute inverses with composition.

1.3.4 Example. Show x has no inverse.

Solution. If $B(x) = b_0 + b_1 x + b_2 x^2 + \cdots$ is any formal power series, then

$$xB(x) = b_0 x + b_1 x^2 + b_2 x^3 + \cdots .$$

If $B(x)$ were the inverse of x, then we would have that $[x^0]xB(x) = 1$, but it is not. Thus x has no inverse. □

Which formal power series have inverses?

1.3.5 Theorem. *A formal power series has an inverse if and only if it has a non-zero constant term.*

Proof. Let $A(x)$ be a formal power series with constant term α.

If $\alpha \neq 0$, then there is a formal power series $C(x)$ such that $A(x) = \alpha(1 - xC(x))$. Let $P(x) = (1 - x)$; thus $A(x) = \alpha P(xC(x))$. We know that $P(x)^{-1} = 1 + x + x^2 + \cdots$ from our example. Moreover, by Composition Theorem $P(xC(x))$ is well-defined, since $xC(x)$ has a zero constant term. Thus $A(x)^{-1} = \alpha^{-1}P(xC(x))^{-1}$ is well-defined formal power series.

If $\alpha = 0$, then $A(x) = xC(x)$ for some formal power series $C(x)$. Note for any formal power series $B(x)$, the product $A(x)B(x) = xC(x)B(x) \neq 1$. Thus $A(x)$ has no inverse. □

1.3.6 Theorem. *If a formal power series has an inverse, then it has a unique inverse.*

Proof. Suppose that $A(x), B_1(x), B_2(x)$ are formal power series where $B_1(x)$ and $B_2(x)$ are both inverse of $A(x)$. Then

$$B_1(x) = B_1(x)(A(x)B_2(x)) = (B_1(x)A(x))B_2(x) = B_2(x)$$

with some help from your assignment on Thursday. □

1.4 *Binomial Theorem Revisited*

For any real number a and non-negative integer k, we can define

$$\binom{a}{k} = \frac{a(a-1)\cdots(a-k+1)}{k!}.$$

1.4.1 Theorem (Binomial Theorem)**.** *For any rational number a,*

$$(1+x)^a = \sum_{k\geq 0}\binom{a}{k}x^k.$$

We will not prove this theorem but we will need to use it. For example,

$$\binom{-n}{k} = \frac{(-n)(-n-1)\cdots(-n-k+1)}{k!} = (-1)^k\binom{n+k-1}{k}.$$

We can use this to write:

$$(1-x)^{-n} = \sum_{k\geq 0}\binom{n+k-1}{k}x^k.$$

1.4.2 Lemma.

$$(1+x)^{1/2} = 1 + \sum_{k\geq 0}\frac{(-1)^{k-1}}{2^{2k-1}k}\binom{2k-2}{k-1}x^k$$

Proof. (Sketch). First we show that

$$\binom{1/2}{k} = \frac{(-1)^{k-1}}{2^{2k-1}k}\binom{2k-2}{k-1},$$

then the result will follow from the Binomial Theorem. □

Let's do a composition of this with $h(x) = -4x$. We will obtain:

$$(1-4x)^{1/2} = 1 - 2\sum_{k\geq 1}\frac{1}{k}\binom{2k-2}{k-1}x^k. \qquad (1.4.1)$$

We will need this later.

1.5 Catalan paths and Catalan numbers

Lattice paths are paths on the integer lattice in 2-dimensions with steps that consist of moving one unit up ("u") and unit right ("r"). What is the number of lattice paths from $(0,0)$ to (m,n)? This number is $\binom{m+n}{n}$.

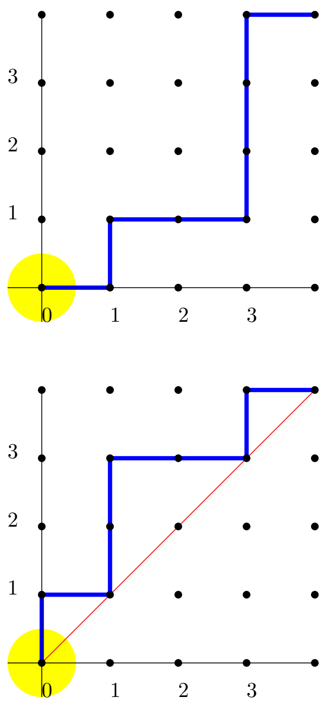

Figure 1.1: Lattice paths and Catalan paths.

Lattice paths: paths starting at $(0,0)$ and takes n up and n right steps.

Catalan paths: paths starting at $(0,0)$, takes n up and n right steps and does not go below the $y = x$ line.

A *Catalan path* is a lattice path from $(0,0)$ to (n,n) which never goes below the $y = x$ line. Here, we will say that a Catalan path has length n if it ends at (n,n). Let c_n be the number of Catalan path of length n. We have

$$c_0 = 1, c_1 = 1, c_2 = 2, c_3 = 5, \ldots$$

Figure 1.2 shows all of the Catalan paths of length 3. Our next step in determining c_n is the find a recurrence.

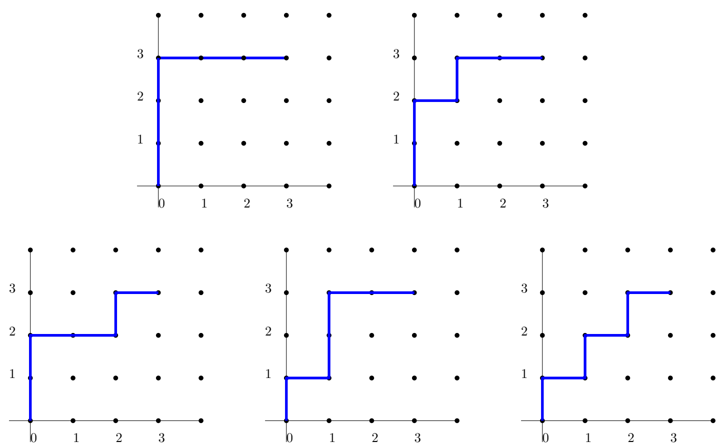

Figure 1.2: All Catalan paths for $n = 3$.

Any Catalan path of positive length touches the line $y = x$ at least twice. If it meets $y = x$ exactly twice, then it can be represented as a sequence of us and rs as follows:

$$u\alpha r$$

where α is a Catalan path of length $n-1$.

1.5.1 Theorem. *If c_n is the number of Catalan paths of length n, then $c_0 = 1$ and if $n \geq 1$,*

$$c_n = \sum_{i=1}^{n} c_{i-1}c_{n-i}.$$

Proof. For $n > 0$, let γ be a Catalan path of length n and let i be the least positive integer such that γ pass through (i, i). Then γ splits into two parts; the first part is a Catalan path γ_1 of length i which only meets the $y = x$ line twice and the second part γ_2 is a Catalan path from (i, i) to (n, n).

The number of choices for γ_1 is c_{i-1} and the number of choices for γ_2 is c_{n-i}. Summing over all possible values of i, we get the required result. □

Next let's look at the generating series. Let $C(x)$ be the generating function for the Catalan paths of length n;

$$C(x) = \sum_{k\geq 0} c_k x^k.$$

Recall that

$$A(x)B(x) = \sum_{n\geq 0}\left(\sum_{k=0}^{n} a_k b_{n-k}\right).x^n$$

1.5.2 Theorem. *If $C(x)$ is the generating function for the Catalan numbers, then*

$$C(x) = 1 + xC(x)^2.$$

Proof. We will consider the coefficients of x^n on both sides. The coefficient of x^0 on both sides is 1.

We may assume $n > 0$. We have $[x^n]C(x) = c_n$ by definition. The coefficient of x^n in $1 + xC(x)^2$ is the coefficient of x^{n-1} in $C(x)^2$, which is

$$\sum_{k=0}^{n-1} c_k c_{n-1-k} = \sum_{i=1}^{n} c_{i-1}c_{n-i}$$

which is equal to c_n by Theorem 1.5.1. □

Now we have $1 - C(x) + xC(x)^2 = 0$.

The set for formal power series forms a ring. A *rational function* is $\frac{P(x)}{Q(x)}$ where $P(x), Q(x)$ are formal power series and $Q(x) \neq 0$. The rational functions form a field.

We have quadratic equation over the field of rational functions, and we can solve it using the usual formula as follows:

$$C(x) = \frac{1 \pm (1-4x)^{1/2}}{2x}.$$

Recalling what we did in equation (1.4.1), we obtain the following:

$$C(x) = \frac{1}{2x} \pm \frac{1}{2x}(1-4x)^{1/2} = \frac{1}{2x} \pm \left(\frac{1}{2x} - \frac{1}{x} \sum_{k \geq 1} \frac{1}{k} \binom{2k-2}{k-1} x^k \right).$$

We know that $C(x)$ is a generating function and so has non-negative coefficients, and so we know we have to take the minus in the $\pm$ coming the quadratic formula to obtain:

$$C(x) = \sum_{k \geq 1} \frac{1}{k} \binom{2k-2}{k-1} x^{k-1} = \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} x^n.$$

1.5.3 Theorem. *The number of Catalan paths of length n is $\frac{1}{n+1}\binom{2n}{n}$.*

In general, this is a method of enumeration; first find a recurrence relation for the coefficients, translate the recurrence into a recurrence for the generating function, solve the equation involving the generating function and then extract the coefficients using the Binomial Theorem.

The Catalan numbers are a very famous sequence of numbers. It is largest entry in OEIS, and they enumerate many combinatorial classes of objects. If S and S' are sets with weight functions w and w', respectively, and the number of elements of weight n in S is equal to the number of elements of weight n in S', then we say that S and S' with their weight functions, are *equinumerous*. (Note that this can happen, even when there is no "combinatorial" bijection.)

We are heading towards enumerating set partitions of $[n]$. We want to count partitions of $[n]$ where every part has distinct size, or partitions where every part has odd size. An *integer partition* of n is a sum $\sum_{i=1}^{m} a_i = n$ where $a_i > 0$ and $a_i \geq a_{i+1}$. Every partition of $[n]$ gives rise to an integer partition of n.

Another class enumerated by the Catalan number is the of "legal" words in brackets. Let $\mathcal{B}$ be the set of strings with symbols "(" and ")", such that the number of (is equal to the number of) and, in any initial string, the number of (is at least the number of) brackets. The weight (or length) of a bracket word is n, if there are n ('s and n)'s. We can give an easy bijection between the set of Catalan paths of length n and the bracket words of length n as follows: consider the map taking (to u and the) to r. This takes bracket words to words in u, r, which we can show are Catalan paths.

1.6 Sum and Product Lemmas

To enumerate the classes of objects that we've discussed, we'll need a few more tools.

1.6.1 Theorem (Sum Lemma)**.** *Let (A, B) be a partition of a set S then*

$$\Phi_S(x) = \Phi_A(x) + \Phi_B(x).$$

Proof. We have that

$$\Phi_S(x) = \sum_{\sigma \in S} x^{w(\sigma)} = \sum_{\sigma \in A} x^{w(\sigma)} + \sum_{\sigma \in B} x^{w(\sigma)} = \Phi_A(x) + \Phi_B(x)$$

since A, B partitions S. □

More generally, if A, B are sets, then

$$\Phi_{A\cup B}(x) = \Phi_A(x) + \Phi_B(x) - \Phi_{A\cap B}(x).$$

For sets A, B, the *cartesian product* of A and B is the set

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

1.6.2 Theorem (Product Lemma)**.** *Let A, B be sets of configuration with weight function α and β, respectively. Let $S = A \times B$. If $w(\sigma) = \alpha(a) + \beta(b)$ for $\sigma = (a, b) \in S$, then*

$$\Phi_{A\times B}(x) = \Phi_A(x)\Phi_B(x).$$

Proof. Exercise. □

1.6.3 Example. Let S be the set of all k-tuples $\sigma = (a_1, \ldots, a_k)$ where a_i is a non-negative integer. Let $w(\sigma) = a_1 + \cdots + a_k$. Determine the generating function.

1.7 *Integer solutions*

Problem: Let k, n be fixed non-negative integers. How many solutions are there to the equation $t_1 + t_2 + \cdots + t_k = n$ when $t_1, t_2, \ldots, t_k$ are non-negative integers?

Solution. If $k = 2$ and $n = 3$, we have

$$3 = 3 + 0 = 0 + 3 = 1 + 2 = 2 + 1$$

four solutions.

Let $N_{\geq i}$ denote the set of integers that are $\geq i$. Let

$$S = N_{\geq 0} \times N_{\geq 0} \times \cdots N_{\geq 0} = N_{\geq 0}^k.$$

Any solution $(t_1, t_2, \ldots, t_k)$ to the equation is an element of S such that $t_1 + t_2 + \cdots + t_k = n$. Let $w(t_1, t_2, \ldots, t_k) = t_1 + t_2 + \cdots + t_k$. The solution to our problem is

$$[x^n]\Phi_S(x).$$

But S is a Cartesian product. If we define $w(t_i) = t_i$ for all $t_i \in N_{\geq 0}$, then we may apply the Product Lemma to obtain,

$$\Phi_S(x) = (\Phi_{N_{\geq 0}}(x))^k.$$

By apply the Sum Lemma, we can obtain

$$\Phi_{N_{\geq 0}}(x) = \sum_{i\geq 0} \Phi_{\{i\}}(x) = \sum_{i \geq 0} x^i.$$

So we have

$$\Phi_S(x) = \left(\sum_{i\geq 0} x^i\right)^k = (1-x)^{-k}.$$

Recall

$$[x^n](1-x)^{-k} = \binom{n+k-1}{n}$$

from the Binomial Theorem. Thus the answer is $\binom{n+k-1}{n}$.

Now that we can do the basics, let's add some restrictions.

Problem: Let k, n be fixed non-negative integers. How many solutions are there to the equation $t_1 + t_2 + \cdots + t_k = n$ when $t_1, t_2, \ldots, t_k$ are non-negative integers such that $t_i \geq i$, for $i = 1, \ldots, k$?

Solution. The generating function for the solution is

$$\begin{aligned}
\Phi_{N_{\geq 1}\times N_{\geq 2}\times\cdots\times N_{\geq k}}(x) &= \prod_{i=1}^{k} \Phi_{N_{\geq i}}(x) \\
&= \prod_{i=1}^{k} \sum_{j\geq i} x^j \\
&= \prod_{i=1}^{k} x^i \sum_{j\geq 0} x^j \\
&= \prod_{i=1}^{k} x^i (1-x)^{-1} \\
&= x^{1+2+\cdots+k}(1-x)^{-k} \\
&= x^{\binom{k+1}{2}}(1-x)^{-k}
\end{aligned}$$

and the answer

$$[x^n]x^{\binom{k+1}{2}}(1-x)^{-k} = [x^{n-\binom{k+1}{2}}](1-x)^{-k} = \binom{n-\binom{k}{2}-1}{n-\binom{k+1}{2}}.$$

Let's check that our Product and Sum Lemmas agree with what we already know. A subset of size k of $[n]$ can be uniquely represented as a vector of $n-k$ 0s and k 1s. Ex. $\{1, 2\}$ of $[5]$ is $(1, 1, 0, 0, 0)$.

Problem: Let k, n be fixed non-negative integers. How many solutions are there to $t_1 + \cdots + t_n = k$, where $t_i \in \{0, 1\}$ for $i = 0, \ldots, n$?

Solution. The generating function for the set of solution is

$$\Phi_{\{0,1\}^n}(x) = \left(\Phi_{\{0,1\}}(x)\right)^n = (1+x)^n$$

whose x^k coefficient is $\binom{n}{k}$, as it should be. □

1.8 *Compositions*

A *composition* of an integer n with k parts is an ordered list $(c_1, \ldots, c_k)$ of positive integers $c_1, \ldots, c_k$ such that $c_1 + \cdots + c_k = n$. Here $n, k \geq 1$, but for mathematical convenience we will allow a single (empty) composition with 0 parts, which is a composition of 0.

For example, the composition of 4 are as follows

$$(4), (3, 1), (2, 2), (1, 3), (2, 1, 1), (1, 2, 1), (1, 1, 2), (1, 1, 1, 1).$$

Problem: How many compositions of n with k parts are there, for $n, k \geq 1$?

Solution. Let $S = N^k_{\geq 1}$ and $w(c_1, \ldots, c_k) = c_1 + \cdots + c_k$. Then, a composition of n with k parts is an element of S with weight n. Let $w(c_i) = c_i$ for $c_i \in N_{\geq i}$, for all i. Then the Product Lemma may be applied to obtain

$$\Phi_S(x) = \Phi_{N^k_{\geq 1}}(x) = \left(\Phi_{N_{\geq 1}}(x)\right)^k.$$

Then,

$$[x^n]\Phi_S(x) = [x^n]\left(\sum_{i\geq 1} x^i\right)^k = [x^n]x^k\left(\sum_{i\geq 0} x^i\right)^k = [x^n]x^k\,(1-x)^{-k}$$

by the Geometric Series. This is the same as $[x^{n-k}]\,(1-x)^{-k}$ which is

$$\binom{k+n-k-1}{n-k} = \binom{n-1}{n-k}. \qquad \square$$

Problem: How many compositions of n with k parts are there, in which each part is an odd number?

Solution. Let $S = N^k_{odd}$ where $N_{odd} = \{1, 3, 5, 7, \ldots\}$. We will use the same weight functions from before. The required compositions are the elements of S with weight n and so it is

$$\begin{aligned}
[x^n]\Phi_S(x) &= [x^n]\Phi_{N^k_{odd}}(x) \\
&= [x^n]\left(\Phi_{N_{odd}}(x)\right)^k, \quad \text{by Product Lemma} \\
&= [x^n]\left(x\sum_{i\geq 0} x^{2i}\right)^k \\
&= [x^n]x^k(1-x^2)^{-k}, \quad \text{by Geometric Series} \\
&= [x^{n-k}](1-x^2)^{-k} \\
&= [x^{n-k}]\sum_{i\geq 0}\binom{k+i-1}{i}x^{2i}.
\end{aligned}$$

The coefficient is 0 is $n-k$ is odd (as it should be). If $n-k$ is even, then the term we want is $i = {}^{n-k}/2$ and we obtain

$$\binom{k+\frac{n-k}{2}-1}{\frac{n-k}{2}} = \binom{\frac{n+k-2}{2}}{\frac{n-k}{2}}.$$

Problem: How many compositions of n are there, with k parts, where each part is at most 5?

Solution. Let $S = \{1, 2, 3, 4, 5\}^k$ and the required compositions are exactly the elements of S with weight n (retaining our weight functions from before). Thus the required number is

$$
\begin{aligned}
[x^n]\Phi_S(x) &= [x^n]\Phi_{\{1,2,3,4,5\}^k}(x) \\
&= [x^n]\left(\Phi_{\{1,2,3,4,5\}}(x)\right)^k, \quad \text{by Product Lemma} \\
&= [x^n]\left(x^1+x^2+x^3+x^4+x^5\right)^k \\
&= [x^n]\left(\frac{x(1-x^5)}{1-x}\right)^k, \quad \text{Geometric Series} \\
&= [x^n]x^k(1-x^5)^k(1-x)^{-k} \\
&= [x^{n-k}]\left(\sum_{i\geq 0}\binom{k}{i}(-x^5)^i\right)\left(\sum_{j\geq 0}\binom{k+j-1}{j}x^j\right) \\
&= [x^{n-k}]\left(\sum_{i\geq 0}\binom{k}{i}(-x^5)^i\right)\left(\sum_{j\geq 0}\binom{k+j-1}{j}x^j\right) \\
&= [x^{n-k}]\sum_{i\geq 0}\sum_{j\geq 0}\binom{k}{i}(-1)^i\binom{k+j-1}{j}x^{5i+j} \\
&= \sum_{i\geq 0, 5i+j=n-k}\quad\sum_{j\geq 0}\binom{k}{i}(-1)^i\binom{k+j-1}{j} \\
&= \sum_{i=0}^{\lfloor\frac{n-k}{5}\rfloor}\binom{k}{i}(-1)^i\binom{n-5i-1}{n-k-5i}.
\end{aligned}
$$

Next, what happens the number of parts is not specified?

Problem: How many composition of n are there, for $n \geq 0$?

Solution. Let $S = \bigcup_{k\geq 0} N_{\geq 1}^k$ where $k = 0$ is the empty composition of 0.

The compositions of n are elements of S of weight n, retaining our weight functions from before, and so the number is

$$
\begin{aligned}
[x^n]\Phi_S(x) &= [x^n]\Phi_{\bigcup_{k\geq 0} N_{\geq 1}^k}(x) \\
&= [x^n]\sum_{k\geq 0}\Phi_{N_{\geq 1}^k}(x), \quad \text{by the Sum Lemma} \\
&= [x^n]\sum_{k\geq 0}(\Phi_{N_{\geq 1}}(x))^k, \quad \text{by the Product Lemma} \\
&= [x^n]\sum_{k\geq 0}(\sum_{i\geq 1}x^i)^k \\
&= [x^n]\sum_{k\geq 0}(x(1-x)^{-1})^k, \quad \text{by Geometric Series} \\
&= [x^n]\frac{1}{1-x(1-x)^{-1}}, \quad \text{by Geometric Series} \\
&= [x^n]\frac{1-x}{1-x-x} \\
&= [x^n]\frac{1-2x+x}{1-2x} = [x^n]1+\frac{x}{1-2x}
\end{aligned}
$$

We use the Geometric Series formula again to obtain

$$\begin{aligned}[x^n]\Phi_S(x) &= [x^n]1 + \frac{x}{1-2x} \\ &= [x^n](1 + x\sum_{i\geq 0}(2x)^i) \\ &= [x^n](1 + \sum_{i\geq 0} 2^i x^{i+1}) \\ &= \begin{cases} 1, & n = 0 \\ 2^{n-1}, & n \geq 1. \end{cases}\end{aligned}$$

We used here that $i + 1 = n$. □

The number of composition, say a_n, of compositions of n in which all parts are at least 3 is

$$a_n = [x^n]\frac{1-x}{1-x-x^3}.$$

What is a_n? Well, in this case, it's a bit messy, so we will instead only give a recurrence for a_n.

Let $A(x) = \sum_{n\geq 0} a_n x^n$. We have

$$\begin{aligned} A(x) &= \frac{1-x}{1-x-x^3} \\ (1-x-x^3)A(x) &= 1-x \\ A(x) - xA(x) - x^3A(x) &= 1-x. \end{aligned}$$

For $n \geq 3$, we see

$$\begin{aligned} [x^n](A(x) - xA(x) - x^3A(x)) &= [x^n](1-x) \\ [x^n]A(x) - [x^{n-1}]A(x) - [x^{n-3}]A(x) &= 0 \\ a_n - a_{n-1} - a_{n-3} &= 0 \\ a_n &= a_{n-1} + a_{n-3}. \end{aligned}$$

We have that $a_0 = 1$, $a_1 = 0$, $a_2 = 0$. The relation that we found is a linear recurrence and the initial conditions.

2

Words and Languages

You may recall that the Catalan path were encoded as strings of u's and r's – in other words, as binary strings. Many useful objects can also be encoded as binary strings and we will now enumerate these object in more detail.

Let Σ denote some set of symbols (often finite); we will refer to Σ as an *alphabet*. A *word* over the alphabet Σ is a finite sequence

$$a_1 a_2 \cdots a_k$$

where $a_i \in \Sigma$. Here, k is the *length* of the word. We will allow the empty word of length 0, which we will denote ϵ. If we are given two words over Σ, say

$$\alpha = a_1 a_2 \cdots a_k, \quad \beta = b_1 \cdots b_\ell$$

then $\alpha\beta$ denotes their *concatenation*, which is the word

$$a_1 a_2 \cdots a_k b_1 \cdots b_\ell.$$

We note $\alpha\epsilon = \epsilon\alpha = \alpha$ and that concatenation is associative

$$(\alpha\beta)\gamma = \alpha(\beta\gamma).$$

This means that the set of all words over Σ under concatenation operation forms a monoid, or a semigroup with an identity.

A *(formal) language* over Σ is a subset of the set of all words over Σ. We denote the set of all words over Σ by Σ^*.

Since a language L are subsets of Σ^*, the complement $\bar{L}$ of L is a language and, if L, M are languages, then so is their union and intersection. We can also define the *product* LM by

$$LM = \{\alpha\beta \mid \alpha \in L, \beta \in M\}.$$

Hence we can define L^n recursively by

$$L^0 = \{\epsilon\}, L^{n+1} = LL^n.$$

We commonly denote the union of L and M by $L + M$, and then we define the *Kleene closure* L^* of L by

$$L^* = L^0 + L^1 + L^2 + \cdots = \sum_{k \geq 0} L^k.$$

This notation is consistent with the definition of Σ^*, which is the Kleene closure of the alphabet.

For example, we will later consider the set of binary strings, which is $\{0,1\}^*$. In this context, we often call the words "strings".

A *weight function on an alphabet* Σ is a function from Σ to the non-negative integers. We may think of a weight function as a rule assigning a weight to each letter. The *weight* $w(\alpha)$ for a word $\alpha = a_1 \cdots a_k$ is

$$w(\alpha) = \sum_{i=1}^{k} w(a_i).$$

Let us consider a familiar example, the Catalan paths, as running example.

Let $\mathcal{L}$ be the language of $\Sigma = \{a, b\}$ that consists of all words that contain n copies of a and n copies of b, and have the property that if $\alpha = \beta\gamma$ then the number of a's in β is at least the number of b's, for $n = 0, 1, 2, \ldots$. If we define $w(a) = 1$ and $w(b) = 0$, the weight of a word will be the number of a's in it.

We will prove

$$\mathcal{L} = \{\epsilon\} + a\mathcal{L}b\mathcal{L}. \tag{2.0.1}$$

First, ϵ is the unique word in $\mathcal{L}$ with 0, and so we need only prove that every non-empty word in $\mathcal{L}$ is in $a\mathcal{L}b\mathcal{L}$. Suppose $\alpha \in \mathcal{L}$ has non-zero weight, then we factorise α as

$$\alpha = \beta\gamma$$

where $\beta, \gamma \in \mathcal{L}$ and $w(\beta) > 0$. There clearly exists at least one such factorisation, and so we will take the unique factorisation such that $w(\beta)$ is minimal. What are the first and last elements of β?

The last element of β must be b and so $\beta = a\beta_1 b$, where $\beta_1 \in L$. So $\beta \in a\mathcal{L}b$ and thus $\alpha \in a\mathcal{L}b\mathcal{L}$. This show that each word α of positive weight in $\mathcal{L}$ can be uniquely factorised as $\alpha \in a\mathcal{L}b\mathcal{L}$.

Now we do some enumeration. We define a map Ψ for languages of $\{a, b\}$ to generating functions. As usual, we set $\Psi(a) = x$ and $\Psi(b) = 1$ and if $\alpha = a_1 \cdots a_k \in \{a, b\}^*$, then

$$\Psi(\alpha) = \prod_{i=1}^{k} \Psi(a_i) = x^{w(\alpha)}.$$

Finally, if $L \subseteq \{a, b\}^*$, then

$$\Psi(L) = \sum_{\alpha \in L} \Psi(\alpha).$$

Then $\Psi(L)$ is the generating function for L with the given weight function.

We immediately see that if M_1 and M_2 are languages of Σ and $M_1 \cap M_2 = \emptyset$, then

$$\Psi(M_1 + M_2) = \Psi(M_1) + \Psi(M_2).$$

2.0.1 Lemma. *Let L, M_1, M_2 be languages of Σ and $L = M_1 M_2$. If each word α in L can be expressed in exactly one way as a concatenation ("uniquely created") $\beta_1\beta_2$ where $\beta_i \in M_i$, then*

$$\Psi(L) = \Psi(M_1)\Psi(M_2).$$

Proof. Left as an exercise. □

2.0.2 Corollary. *Suppose L, M are language over Σ and that $L = M^k$. If each word in L can be expressed in exactly one way as the concatenation of k words in M, then $\Psi(L) = \Psi(M)^k$.*

Note that if $\epsilon \in M$, then M^k will not have the unique factorisation property.

We don't have this issue of $\mathcal{L}$, so we may apply the lemma and obtain the following:

$$\Psi(\mathcal{L}) = \Psi(\{\epsilon\}) + \Psi(a)\Psi(\mathcal{L})\Psi(b)\Psi(\mathcal{L}) = 1 + x\Psi(\mathcal{L})^2.$$

Thus we have obtained the equation satisfied by the generating function of the Catalan numbers without first determining the recurrence explicitly.

Compositions are also languages. What is the generating function of the number of ways we can write n as the sum of integers, each of which is 1 or 2? Take $\Sigma = \{a_1, a_2\}$ where $w(a_i) = i$. Then Σ^k will give the number of ways of writing n as k 1s and 2s and so

$$\Psi(\Sigma^*) = \sum_{k\geq 0}(x + x^2)^k = \frac{1}{1-(x+x^2)} = \frac{1}{1-\Psi(\Sigma)}.$$

In fact, we can formalize this more generally as follows.

2.0.3 Theorem. *Let w be a weight function of Σ and let L be a language over Σ such that*

1. *if $i \neq j$, then $L^i \cap L^j = \emptyset$;*
2. *if $\alpha \in L^k$, then there is a unique choice of elements $\beta_1, \ldots, \beta_k$ in L such that $\alpha = \beta_1 \ldots b_k$.*

Then

$$\Psi(L^*) = \frac{1}{1-\Psi(L)}.$$

Proof. $L^* = \{\epsilon\} + L^1 + L^2 + \cdots$. The sets in the sum are pairwise disjoint by condition 1 and each element of L^k can be expressed uniquely, by condition 2, and so

$$\Psi(L^*) = \sum_{k\geq 0}\Psi(L^k) = \sum_{k\geq 0}\Psi(L)^k = \frac{1}{1-\Psi(L)}$$

as required. □

The set of all binary strings is $\{0,1\}^*$. Consider this language with weight function $w(0) = 1, w(1) = 1$. Clearly all the strings over the alphabet $\{0,1\}$ with weight m are uniquely obtained from $\{0,1\}^m$, as a Cartesian product, by "removing all the commas". Thus they are uniquely generated and

$$\Psi(\{0,1\}^*) = \frac{1}{1-\Psi(\{0,1\})} = \frac{1}{1-(2x))} = \sum_{i\geq 0} 2^i x^i$$

which is what is ought to be.

2.0.4 Lemma. $\{0,1\}^* = \{1\}^*(\{0\}\{1\}^*)^*$ *and the element of* $\{0,1\}^*$ *are uniquely created on the right hand side of this equality.*

Proof. Each element α of $\{0,1\}^*$ has k occurrences of 0, for some k. Thus,

$$\alpha = b_1 0 b_2 0 \cdots b_k 0 b_{k+1}$$

where b_i are (possibly empty) strings consisting entirely of 1s. Thus $b_i \in \{1\}^*$, and the result follows. □

Problem: Let a_n be the number of words in $\{0,1\}^*$ which contain no consecutive pair of 1s. Show that

$$a_n = [x^n]\frac{1+x}{1-x-x^2}$$

for $n \geq 0$.

Solution. Let S be the set of words in $\{0,1\}^*$ which contain no consecutive pair of 1s. We see that

$$\{0,1\}^* = \{1\}^*(\{0\}\{1\}^*)^* = \{\epsilon, 1, 11, \ldots\}(\{0\}\{\epsilon, 1, 11, \ldots\})^*.$$

Then,

$$S = \{\epsilon, 1\}(\{0\}\{\epsilon, 1\})^*$$

which uniquely creates the elements of S. And now

$$\Psi(S) = \Psi(\{\epsilon, 1\})\Psi((\{0\}\{\epsilon, 1\})^*) = (1+x)(1-x(1+x))^{-1}$$

and the result follows.

2.1 *Forbidden substrings*

If L is a language, we use L^+ to denote the set of non-empty words in L^*. Then, if ϵ is not in L, then $L^+ = LL^*$ and so

$$\Psi(L^+) = \frac{\Psi(L)}{1-\Psi(L)}.$$

Using our block decompositions from previous class, we can now count words with prescriptions on the lengths of the blocks. What if we want to forbid a given substring?

Let $\Sigma = \{a, b\}$ and let L be the set of word over Σ that do not contain the substring aba. We want to know the number of words in L of length n, for each n.

To do this, we need an auxiliary language. Let M denote the set of words of Σ which have form γaba and contain exactly one copy of aba (so the final copy is the only one).

Now $M \neq Laba$. Why? Some words in L can end in ab. What is $Laba$? If $\alpha \in L$ ends in a or bb, then $\alpha aba \in M$. Otherwise, α ends in ab and so $\alpha = \gamma ab$, then $\gamma ababa \in Mba$. Thus

$$Laba = M + Mba. \tag{2.1.1}$$

Also, consider $L + M$. We can show

$$\epsilon + La + Lb = L + M \tag{2.1.2}$$

Let $L(x)$ and $M(x)$ be the generating function $\Psi(L)$ and $\Psi(M)$, respectively. Then, from (1) and (2), we get

$$\begin{aligned} x^3 L(x) &= M(x) + x^2 M(x) \\ 1 + xL(x) + xL(x) &= L(x) + M(x). \end{aligned}$$

We can just solve this system, like any other system of two equations in L, M. We get

$$\begin{aligned} M(x) &= 1 + (2x-1)L(x) \\ x^3 L(x) &= (1 + (2x-1)L(x))(1+x^2) \end{aligned}$$

Then we can obtain

$$L(x) = \frac{1+x^2}{1-2x+x^2-x^3}$$

and

$$M(x) = \frac{x^3}{1-2x+x^2-x^3}.$$

2.1.1 *A word game*

Let's play a game. We toss a fair coin with sides labelled a and b, stopping once we have observed the sequence aba. What is the average length of the game?

2.1.2 *Probability*

Suppose we have some "random variable" taking non-negative integer values, where p_i denotes the probability that it takes the value i. So $p_i \geq 0$ and $\sum_i p_i = 1$. The *probability generating function* $P(x)$ is the following:

$$P(x) = \sum_{n \neq 0} p_n x^n.$$

For example, if our random variable is the result of fair coin toss, then

$$P(x) = \frac{1}{2} + \frac{1}{2}x.$$

If the random variable is the number of times head occurs when we toss the coin n times, then

$$P(x) = \left(\frac{1}{2} + \frac{1}{2}x\right)^n = \sum_{k=0}^{n} 2^{-n} \binom{n}{k} x^k.$$

2.1.1 Theorem. *If we have independent events with probability generating functions $P(x)$ and $Q(x)$, then the probability generating function of their sum is $P(x)Q(x)$.* □

2.1.2 Lemma. *If $P(x)$ is the probability generating function of a random variable, then $P'(1)$ is the average values of the random variable.* □

Note that the series of $P'(1)$ might not converge; in that case, we would say the expectation is infinite.

Back to our game. The probability that we stop on the nth coin toss is 2^{-n} times the number of strings in M with weight n. If

$$M(x) = \sum_i c_i x^i$$

then this probability is $c_n/2^n$. Then we see,

$$\sum_i c_i 2^{-i} = M(1/2) = 1$$

as we would expect. The expected length of the game will be $1/2M'(1/2) = 10$.

2.1.3 *The game, revisited*

Let's play another game. Two players toss a fair coin consecutively, recording heads as a and tails as b. Before starting, they each choose distinct words α and β of length k in $\{a, b\}^*$. The player whose word appears first wins.

Problem: Given α, β, determine the probability that the player who chooses α wins.

To solve this, we need some machinery. A *prefix* of α is an initial substring of α. The word *abaa* has 5 prefixes: $\epsilon, a, ab, aba, abaa$. A *suffix* of α consists of the last i symbols on α. We define the set of *quotients* $\beta : \alpha$ of α and β to be the set of suffixes σ of β such that $\beta = \pi\sigma$ and π is a non-empty suffix of α. So there is one element of $\beta : \alpha$ arising for each prefix of β that is also a suffix of α. Let's see an example. If $\alpha = abba$ and $\beta = baba$, then

$$\beta : \alpha = \{ba\}, \quad \alpha : \beta = \{bba\}.$$

We can see that $\beta : \alpha$ is a finite language and $\epsilon \in \beta : \alpha$ iff $\alpha = \beta$.

2.1.3 Lemma. *Let Σ be a finite alphabet and suppose α is a non-empty word over Σ. Let L denote the set of words in Σ^* that do not contain α, and let M denote the set of words in Σ^* of the form $\gamma\alpha$ that contain exactly one copy of α. Then, $L\alpha = M\alpha : \alpha$.*

Proof. Exercise (but exactly the same as for *aba*). □

2.1.4 Theorem. *Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ be a set of words such that no word is a substring of another. Let L denote the set of words that contain no word from S. Let M_i denote the set of words in Σ^* of the form $\gamma\alpha_i$ that contain exactly one copy of α_i, and also no copy of α_j if $j \neq i$. Then*

$$\epsilon + L\Sigma = L + M_1 + \cdots + M_k$$

and

$$L\alpha_i = M_1\alpha_i : \alpha_1 + M_2\alpha_i : \alpha_2 + \cdots + M_k\alpha_i : \alpha_k$$

for $i = 1, \ldots, k$. □

We will illustrate this with an example. Let $\Sigma = \{a, b\}$ and

$$\beta = aaa, \quad \gamma = bba.$$

Let B be the set of words in Σ^* that contain exactly one copy of β, as a suffix, and also no copy of γ and let C the set of words in Σ^* that contain exactly one copy of γ, as a suffix, and also no copy of β. We compute

$$aaa : aaa = \{\epsilon, a, aa\}, \quad bba : aaa = \emptyset$$

and

$$aaa : bba = \{aa\}, \quad bba : bba = \{\epsilon\}.$$

Let L be the set of words that contain no copies of β or γ. Then, we have the following:

$$\begin{aligned} \epsilon + La + Lb &= L + B + C \\ Laaa &= B\{\epsilon + a + aa\} + Caa \\ Lbba &= C \end{aligned}$$

Let $L(x), B(x)$ and $C(x)$ be the generating function of L, B, C, respectively.

$$\begin{aligned} 1 + 2xL(x) &= L(x) + B(x) + C(x) \\ x^3L(x) &= B(x)(1 + x + x^2) + x^2C(x) \\ x^3L(x) &= C(x). \end{aligned}$$

We can solve this to obtain the following:

$$B(x) = \frac{x^3 - x^5}{1 + x + x^2} L(x)$$

and also the rational function for L. Now $B(1/2)$ is the probability that β occurs before γ and $C(1/2)$ is the probability that γ occurs before β. And indeed, $L(1/2) = 28/5$ and so $B(1/2) = 3/10$.

Question: if you get to choose first which 3 letter word, which do you choose?

2.2 *Partitions of an integer*

A partition of an integer n with k parts is an expression

$$n = a_1 + a_2 + \cdots + a_k$$

where $a_1, \ldots, a_k$ are positive integers and $a_i \geq a_{i+1}$ for $i < k$. If π denotes a partition, then $|\pi|$ is the sum of its parts. We want to know the number of partitions of n with any number of parts; this is denoted by $p(n)$.

Ex.

$$1 = 1;$$

$$2 = 2 = 1 + 1;$$

$$3 = 3 = 2 + 1 = 1 + 1 + 1$$

$$4 = 4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$$

so we wee that $p(1) = 1$, $p(2) = 2$, $p(3) = 3$ and $p(4) = 5$.

Let $\Sigma = \{a_1, a_2, \ldots\}$ where $w(a_i) = i$ for $i = 1, 2, \ldots$. If L is the language given by

$$L = \{a_1\}^*\{a_2\}^*\{a_3\}^* \cdots$$

then there is a bijection from the elements of L of weight n to the partitions of n. Thus the generating function is

$$\frac{1}{1-x}\frac{1}{1-x^2}\frac{1}{1-x^3}\cdots = \prod_{n\geq 1}\frac{1}{1-x^n}.$$

(Trust me for now that this is a formal power series).

We want to show that number of partitions of n with only odd parts and the number of partitions of n where all parts are distinct, are equinumerous.

The generating function of the number of partitions of n with only odd parts is the generating function for

$$\{a_1\}^*\{a_3\}^*\{a_5\}^* \cdots$$

and so it is

$$\prod_{n\geq 1}\frac{1}{1-x^{2n-1}}.$$

The generating function for partitions of n where all parts are distinct is the generating series of

$$D = (\epsilon + a_1)(\epsilon + a_2)(\epsilon + a_3)\cdots$$

and so the generating function is

$$\prod_{n\geq 1}(1+x^n).$$

2.2.1 Lemma. *The number of partitions of n with only odd parts is equal to the number of partitions of n where all parts are distinct.*

Proof. We have the generating function for partitions of n with only odd parts is equal to

$$\prod_{n\geq 1}\frac{1}{1-x^{2n-1}} = \frac{\prod_{n\geq 1}(1-x^{2n})}{\prod_{n\geq 1}(1-x^n)}$$

since $\prod_{n\geq 1}(1-x^{2n-1})\prod_{n\geq 1}(1-x^{2n}) = \prod_{n\geq 1}(1-x^n)$. Thus

$$\prod_{n\geq 1}\frac{1-x^{2n}}{1-x^n} = \prod_{n\geq 1}\frac{(1+x^n)(1-x^n)}{1-x^n} = \prod_{n\geq 1}(1+x^n).$$

Since the generating functions are equal, the two classes are equinumerous. □

2.3 Exp, log and diff

We define the exponential series $\exp(x)$ by

$$\exp(x) := \sum_{n\geq 0}\frac{x^n}{n!}.$$

It is easy to verify that

$$\exp x + y = \exp x \exp y.$$

Similarly, we define the logarthmic series as follows:

$$\log(1+x) := \sum_{n\geq 1} (-1)^{n-1} \frac{x^n}{n}.$$

Note that if $F(x)$ is a formal power series, the expressions $\exp(F(x))$ and $\log(1 + F(x))$ are well defined iff $F(0) = 0$.

We say that G is a *compositional inverse* of F if $(F \circ G)(x) = x$. Recall that we defined the derivative (of a formal power series) and that the chain rule holds:

$$\frac{d}{dx}(F \circ G)(x) = G'(x)(F' \circ G)(x).$$

Now you can confirm that

$$\frac{d}{dx} \exp(x) = \exp(x)$$

and

$$\frac{d}{dx} \log(1+x) = \frac{1}{1+x}.$$

We will now prove $\exp(\log(1 + x)) = 1 + x$. Let $F(x) = \exp(\log(1 + x))$. Then

$$F'(x) = \frac{1}{1+x} F(x).$$

If the coefficient of x^n in $F(x)$ is f_n, then

$$\begin{aligned} f_n &= [x^n]F(x) \\ &= [x^n](1+x)F'(x) \\ &= [x^n]F'(x) + [x^{n-1}]F'(x) \\ &= (n+1)f_{n+1} + nf_n. \end{aligned}$$

Thus $(n + 1)f_{n+1} = (1 - n)f_n$ for all $n \geq 0$. Since $f_0 = 1$, we can deduce that $f_1 = 1$ and $f_2 = 0$. This shows that $f_n = 0$ for $n \geq 2$. Thus $F(x) = 1 + x$.

Now we can "fix" our lack of rigour in the previous calss as follows. If $A_i(x)$ is a formal power series with a constant term 1, for $i \geq 0$, we can define

$$\prod_{i\geq 0} A_i(x) = \exp\left(\sum_{i\geq 0} \log(A_i(x))\right).$$

So the generating function of integer partitions is

$$\prod_{n\geq 1} \frac{1}{1-x^n}$$

is now well-defined.

Also we can write

$$(1+x)^a = \exp(a \log(1+x))$$

for any real number a. And you can prove the Binomial Theorem.

2.4 Fibonacci numbers

The Fibonacci numbers f_n are defined by initial conditions $f_0 = f_1 = 1$ and the linear recurrence

$$f_{n+1} = f_n + f_{n-1}$$

for $n \geq 1$. Let F_n be the element of $\mathbb{R}^2$ defined by

$$F_n = \begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix}$$

and we define the series $F(t)$ by

$$F(t) = \sum_{n\geq 0} t^n F_n.$$

We can view $F(t)$ as a vector with formal power series as entries, or as a formal power series over $\mathbb{R}^2$.

We note that

$$F_{n+1} = \begin{pmatrix} f_{n+2} \\ f_{n+1} \end{pmatrix} = \begin{pmatrix} f_{n+1} + f_n \\ f_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} F_n.$$

whence

$$F_n = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n F_0$$

and

$$F(t) = \sum_{n\geq 0} t^n \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

If we set $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ then

$$\begin{aligned} \sum_{n\geq 0} t^n \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n &= \sum_{n\geq 0} t^n A^n \\ &= (I - tA)^{-1} \\ &= \begin{pmatrix} 1-t & -t \\ -t & 1 \end{pmatrix}^{-1} \\ &= \frac{1}{1-t-t^2} \begin{pmatrix} 1 & t \\ t & 1-t \end{pmatrix}. \end{aligned}$$

since

$$\det \begin{pmatrix} 1-t & -t \\ -t & 1 \end{pmatrix} = 1 - t - t^2.$$

We have

$$F(t) = \frac{1}{1-t-t^2} \begin{pmatrix} 1 & t \\ t & 1-t \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{1-t-t^2} \begin{pmatrix} 1+t \\ 1 \end{pmatrix}$$

and we can conclude that

$$\sum_{n\geq 0} f_n t^n = \frac{1}{1-t-t^2}.$$

3 More counting

3.1 Principle of Inclusion and Exclusion

Let S be the set and A, B be subsets of S. How many elements are there in $S \setminus \{A \cup B\}$? If A, B are disjoint, it is

$$|S| - |A| - |B|$$

but this is wrong when A, B are not disjoint, because the elements in $A \cap B$ have been substracted twice. Instead, it is

$$|S \setminus \{A \cup B\}| = |S| - |A| - |B| + |A \cap B|.$$

Similarly, if A, B, C are subsets of S, we can find that

$$\begin{aligned} |S \setminus \{A \cup B \cup C\}| = |S| &- |A| - |B| - |C| \\ &+ |A \cap B| + |A \cap C| + |C \cap B| \\ &- |A \cap B \cap C|. \end{aligned}$$

We can do this more generally for any number of subsets.

Let S be a set of size N. Let $E_1, \ldots, E_r$ be subsets of S, not necessarily distinct. For any subset M of $\{1, \ldots, r\}$, let $N(M)$ be the following:

$$N(M) := |\cap_{i \in M} E_i|$$

and, for $0 \leq j \leq r$, define

$$N_j := \sum_{|M|=j} N(M).$$

Let $N_0 = N = |S|$.

3.1.1 Theorem (Inclusion/Exclusion). *The number of elements of S not in any of the subsets $E_1, \ldots, E_r$ is*

$$N - N_1 + N_2 - N_3 + \cdots + (-1)^r N_r = \sum_{i=0}^{r} (-1)^i N_i.$$

Proof. If $x \in S$ and x is is none of the E_i, then x contributes 1 to the sum.

If $s \in S$ is in k of the E_i, then x contributes

$$1 - \binom{k}{1} + \binom{k}{2} - \cdots + (-1)^k \binom{k}{k} = (1-1)^k = 0$$

by the Binomial Theorem.

Recall the following problem from your assignment. Let n be a positive integer. Suppose that the prime decomposition of n is

$$n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$$

where $\{p_i\}_{i=1}^r$ are distinct primes and $\{a_i\}_{i=1}^r$ are positive integers. The *Euler function* is the number of integers k with $1 \le k \le n$ such that $\gcd(n,k) = 1$. You showed that:

(a)

$$\phi(n) = n \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right);$$

(b)

$$\sum_{d|n} \phi(d) = n.$$

This brings us to a very classical object in the field of combinatorics is the *Möbius function*, defined as below,

$$\mu(d) = \begin{cases} 0, & \text{if } d \text{ is not square-free;} \\ 1, & \text{if } d \text{ has an even number of prime divisors; and} \\ -1, & \text{if } d \text{ has an odd number of prime divisors.} \end{cases}$$

3.1.2 Theorem. $\sum_{d|n} \mu(d) = \begin{cases} 1, & \textit{if } n = 1; \\ 0, & \textit{else.} \end{cases}$

Proof. For $n = 1$, this is clearly true. Now suppose $n = p_1^{a_1} \cdots p_r^{a_r}$, then

$$\sum_{d|n} \mu(d) = \sum_{i=0}^{r} \binom{r}{i} (-1)^i = (1-1)^r = 0.$$

□

Now we can combine all of this to obtain:

$$\sum_{d|n} \frac{\mu(d)}{d} = \frac{\phi(n)}{n}.$$

3.1.3 Theorem (Möbius Inversion)**.** *Let $f(n)$ and $g(n)$ be function defined for every positive integer n satisfying*

$$f(n) = \sum_{d|n} g(d).$$

Then g satisfies

$$g(n) = \sum_{d|n} \mu(d) f(n/d).$$

Proof. We see that

$$\begin{aligned} \sum_{d|n} \mu(d) f(n/d) &= \sum_{d|n} \mu(n/d) f(d) \\ &= \sum_{d|n} \mu(n/d) \sum_{d'|d} g(d') \\ &= \sum_{d'|n} g(d') \sum_{m|(n/d')} \mu(m) \end{aligned}$$

The inner sum of the final right hand side is 1 or 0; in particular it is 0 unless $d' = n$. □

Why is it called the Möbius inversion? Make a matrix M with rows and columns indexed by positive integers dividing n where

$$M(d_1, d_2) = \begin{cases} 1, & \text{if } d_1 | d_2; \\ 0, & \text{else.} \end{cases}$$

What is on the diagonal of M? All ones. Below the diagonal? All 0s. Thus M is some upper triangular matrix with ones on the diagonal, and is thus invertible. There exists some M^{-1} matrix,

$$M^{-1}(d_1, d_2) = \mu(\frac{d_2}{d_1})$$

where $d_1 | d_2$, and 0 otherwise. If we consider f, g to be vectors, the theorem would that if $g = Mf$, then $f = M^{-1}g$.

Problem: what is the number N_n of circular sequences of 0s and 1s, when two sequences obtained by a rotation are considered the same?

Solution. Let $M(d)$ be the number of circular sequence of length d which are not periodic. Then

$$N_n = \sum_{d|n} M(d).$$

We observe that

$$\sum_{d|n} dM(d) = 2^n$$

because this is the number of all possible circular sequences. By Möbius Inversion, we obtain

$$N_n = \sum_{d|n} M(d) = \sum_{d|n} \frac{1}{d} \sum_{\ell|d} \mu(d/\ell) 2^\ell = \frac{1}{n} \sum_{\ell|n} \phi(n/\ell) 2^\ell$$

after some simplification.

This still looks mysterious, but at least the right hand side consists of all positive terms and we can bound it by some method (but we will not).

3.2 *Plane trees*

A plane tree is a tree drawn in the plane, where there are "levels" and each vertex has a certain number of children in the next level. We have drawn all the plane tree with at most 3 edges in Figure 3.1.

A plane tree is *planted* if its root vertex has only 1 child. Let $T(x)$ denote the generating function for the plane trees, weighted by the number of edges. Using the diagram, we see

$$T(x) = 1 + x + 2x^2 + 5x^3 + \cdots$$

A plane tree with at least one edge decomposes uniquely into a left planted plane tree and another plane tree, as shown figure.

Note that a planted plane tree is obtained by taking a plane tree and adding one vertex adjacent to its root vertex, in the level under

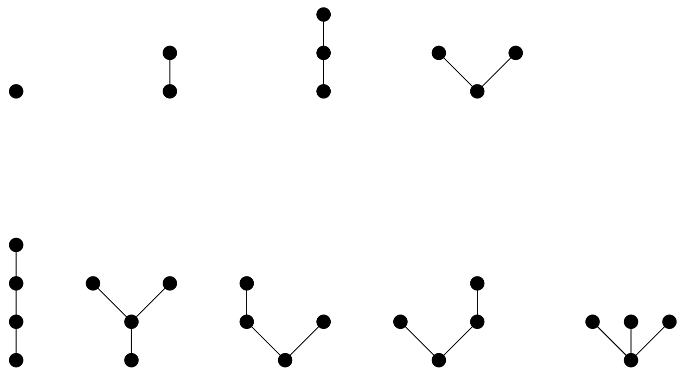

Figure 3.1: All plane trees with at most 3 edges.

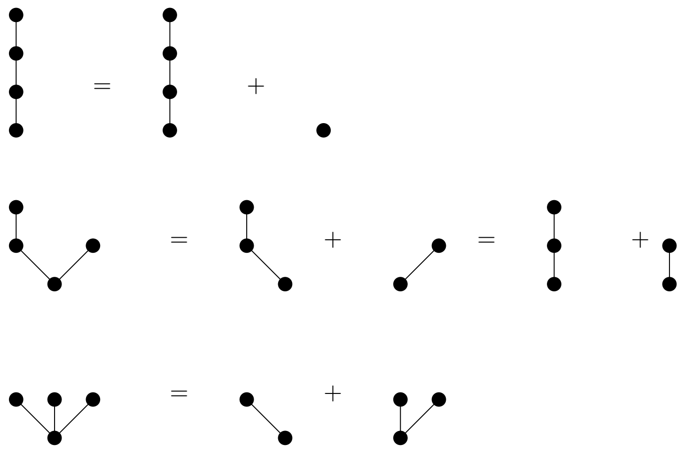

Figure 3.2: Decomposition of a plane tree into a planted plane tree and a plane tree.

it. The class of planted plane trees with n edges is equinumerous with class of plane trees with $n-1$ edges. Thus

$$T(x) = 1 + (xT(x))T(x) = 1 + xT(x)^2$$

and so $T(x)$ satisfies the same equation as $C(x)$. We have shown that the number of plane trees with n edges is the Catalan number c_n.

3.3 *Multivariate Series*

So far, we have counted objects weighted by some integer i. Sometimes it's not enough information. In particular, if we consider partitions of integer n, we might also consider the number of parts, or the largest, as well as n.

For example, the coefficient $t^k x^n$ in

$$\prod_{i\geq 1}(1-tx^i)^{-1} = \prod_{i\geq 1}\left(\sum_{j\geq 0}(tx^i)^j\right)$$

is equal to the number of partition of n with exactly k parts. This is a generating function in 2 variables.

A series

$$\sum_{i,j\geq 0} c_{i,j}x^i t^j = \sum_{j\geq 0}\left(\sum_{i\geq 0} c_{i,j}x^i\right) t^j$$

is a formal power series in variable t with coefficients from the ring of formal series in variable x.

Recall the q-binomial coefficients from your assignment. Let q be a variable. If n is a non-negative integer, define $[n]$ by

$$[n] := \frac{q^n - 1}{q - 1},$$

and define $[n]!$ by $[0]! = 1$ and

$$[n+1]! = [n+1][n]!.$$

Finally define

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]![n-k]!}.$$

(Note that $\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ n-k \end{bmatrix}$.) We call $[n]!$ the *q-factorial function* and $\begin{bmatrix} n \\ k \end{bmatrix}$ the *q-binomial coefficient*.) You showed

(a) Prove that if $n, k \geq 1$, then

$$\begin{bmatrix} n \\ k \end{bmatrix} = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

(b) Prove that if $n, k \geq 1$, then

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k \end{bmatrix} + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

Then, you deduced from parts (a) and (b) that $\begin{bmatrix} n \\ k \end{bmatrix}$ is a polynomial in variable q with non-negative integer coefficients. The series $\begin{bmatrix} n \\ k \end{bmatrix}$ is the generating function (in variable q) for the number of partitions of n with at most k parts, where each part has size at most $n - k$.

3.3.1 Lemma. *We have*

$$\prod_{r \geq 0} \frac{1 - atx^r}{1 - tx^r} = \sum_{n \geq 0} \left(\prod_{k=1}^{n} \frac{1 - ax^k}{1 - x^k} \right) t^n$$

Proof. Let $F(t)$ be the left side of the equation and suppose

$$F(t) = \sum_{n \geq 0} c_n t^n.$$

Then $c_0 = 1$ and

$$F(xt) = \frac{1-t}{1-at} F(t).$$

Hence

$$(1 - at)F(xt) = (1 - t)F(t)$$

and we can find the coefficient of t^n on both sides to obtain

$$x^n c_n - a x^{n-1} c_{n-1} = c_n - c_{n-1}.$$

Thus we can obtain

$$c_n = \prod_{k=1}^{n} \frac{1 - ax^k}{1 - x^k}$$

as required. □

If we substitute $a = 0$ and xt for t, we obtain:

$$\sum_{r\geq 1}(1-tx^r)^{-1} = \sum_{n\geq 0} t^n x^n \prod_{k=1}^{n}(1-x^k)^{-1}.$$

Thus

$$x^n \prod_{k=1}^{n}(1-x^k)^{-1}$$

is the generating series for the partitions of integers with exactly k parts, weighted by the sum of their parts.

3.4 *Ferrer's diagrams*

Take a partition of an integer. Draw a picture; each row corresponds to a part of the partition, where the number of dots is the size of the part, and the largest parts come first.

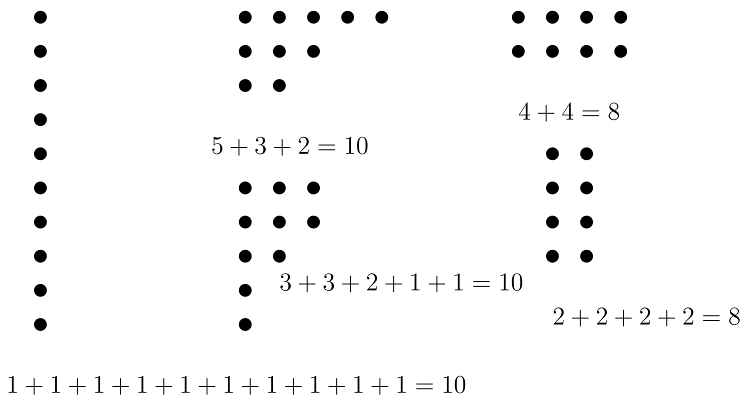

Figure 3.3: Examples of Ferrer's diagrams and their partitions.

The number of Ferrer's diagrams with n dots is equal to the number of partitions of n.

If we transpose a Ferrer's diagram, we get another Ferrer's diagram, called the *conjugate*. We also say that the corresponding partitions are conjugate.

3.4.1 Lemma. *The number of partitions of n with largest part k is equal to the number of partitions of n with exactly k parts.*

Proof. Take the conjugate. □

The generating series for partitions with largest part equal to k, weighted by the sum of the parts is

$$\frac{x^k}{1-x^k}\prod_{i=1}^{k-1}\frac{1}{1-x^i}$$

and the generating series for partitions with exactly k parts is

$$x^k \prod_{r=1}^{k}(1-x^r)^{-1}$$

which are indeed equal.

4
q-analogues

Let q be a variable. If n is a non-negative integer, define $[n]_q$ by

$$[n]_q := \frac{q^n - 1}{q - 1},$$

and define $[n]_q!$ by $[0]_q! = 1$ and

$$[n+1]_q! = [n+1]_q [n]_q!.$$

Finally define

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

(Note that $\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q$.) We call $[n]!$ the *q-factorial function* and $\begin{bmatrix} n \\ k \end{bmatrix}$ the *q-binomial coefficient*.) If q is clear from the context, we will write $[n]$, $[n]!$ and $\begin{bmatrix} n \\ k \end{bmatrix}$ as needed.

For example, the q-bracket $[n]$ is

$$[n]_q = \frac{q^n - 1}{q - 1} = \begin{bmatrix} n \\ 1 \end{bmatrix}_q.$$

Then we can continue:

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q = 1 + q$$

and in general

$$\begin{bmatrix} n \\ 1 \end{bmatrix}_q = 1 + q + \cdots + q^{n-1}.$$

And

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \frac{[4][3]}{[2]} = 1 + q + 2q^2 + q^3 + q^4.$$

4.0.1 Lemma. *We have that*

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} = q^{-k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_q.$$

4.1 q-commuting variables

A *(formal) Laurent series* is a series

$$A(x) = \sum_{n \in \mathbb{Z}} a_n x^n$$

where only finitely many of the coefficients with negative indices is non-zero. The *order* of $A(x)$ is a the greatest integer k such that $x^{-k}A(x)$ is a power series. (Then power series are Laurent series of non-negative order).

Let $\mathbb{F}$ denote the field $\mathbb{R}(q)$ of real Laurent series in variable q. We will work with Laurent series in variable t with coefficients in $\mathbb{F}$; we will view this set of series as a vector space of $\mathbb{F}$. (It is also a field, but this will not matter to us.)

4.1.1 Theorem. *If A, B are operator such that $BA = qAB$ and q commutes with A and B, then*

$$(A+B)^n = \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} A^k B^{n-k}.$$

We say that A, B are *q-commuting variables*. We might ask if such operator actually exist. If

$$C(t) = \sum_n c_n t^n$$

then Q be the linear operator that sends $C(t)$ to $C(qt)$;

$$Q(C(t)) = \sum_n q^n c_n t^n.$$

Let M_t be the operation of multiplication by t. Thus $M_t(C(t)) = tC(t)$. We see that

$$QM_t(F(t)) = Q(tF(t)) = qtF(qt)$$

and

$$M_t Q(F(t)) = tF(qt).$$

Thus it follows that Q and M_t are q-commuting; $QM_t = qM_tQ$.

Now let's apply our theorem; $A = M_tQ$ and $B = Q$. Then

$$A^k B^{n-k}(1) = A^k(1) = q^{\binom{k}{2}} t^k$$

and

$$(A+B)^n(1) = (A+B)^{n-1}(t+1) = (1+t)(1+qt)\cdots(1+q^{n-1}t).$$

This gives us something that we will can the *q-binomial identity*.

4.1.2 Corollary. *We have*

$$\prod_{i=0}^{n-1}(1+q^i t) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} t^k.$$

This suggests that $\prod_{i=0}^{n-1}(1+q^i t)$ is the q-analogue of the power series $(1+t)^n$. We will use the notation

$$\pi(t;n) = \prod_{i=0}^{n-1}(1+q^i t)$$

or $\pi_q(t;n)$ when q is unclear from context. Note, this is not standard notation.

4.2 *q-Differentiation*

If $f(t)$ is a polynomial in t (over $\mathbb{F}$), then we define the operator D_q by

$$D_q(f(t)) = \frac{f(qt) - f(t)}{qt - t}.$$

We call D_q the q-derivative. If $F(t)$ is a Laurent series, we define $D_q(F(t))$ to be the series we get by taking the q-derivative term-by-term.

We note that if $n \geq 1$, then

$$D_q(t^n) = \frac{(qt)^n - t^n}{qt - t} = \frac{t^n}{t}\frac{q^n - 1}{q - 1} = t^{n-1}[n]_q$$

and $D_q(1) = 0$. For fun, let us compute

$$\begin{aligned} D_q(\pi(t;n)) &= \frac{\pi(qt;n) - \pi(t;n)}{qt - t} \\ &= \frac{\pi(qt;n-1)(q^n t - t)}{qt - t} \\ &= [n]\pi(qt;n-1). \end{aligned}$$

We can also compute

$$D_q(\pi(-t;n)^{-1}) = [n]\pi(-t;n+1)^{-1}$$

as expected.

4.2.1 Lemma. *We have that*

$$\prod_{i=0}^{n-1} \frac{1}{1 - q^i t} = \sum_{j \geq 0} \begin{bmatrix} n+j-1 \\ j \end{bmatrix} t^j.$$

Proof. Exercise; use D_q and what we derived in that last class to find a recurrence for the coefficient of t^n on the LHS and deduce that it is a q-binomial coefficient, as advertised. □

Now let us re-index, letting $\ell = n - 1$, to get

$$\prod_{i=0}^{\ell} \frac{1}{1 - q^i t} = \sum_{j \geq 0} \begin{bmatrix} \ell + j \\ j \end{bmatrix} t^j.$$

What is the LHS counting? It is the generating function for the number of partitions of largest part at most ℓ, weighted by the sum of the parts and the size of the largest part.

4.3 *q-Exponentials*

We define the *q-exponential series* $\exp_q(t)$ is given by

$$\exp_q(t) = \sum_{n \geq 0} \frac{t^n}{[n]_q!}.$$

We see immediately that

$$D_q(\exp_q(t)) = \exp_q(t)$$

in analogy with the usual exponentiation.

There are many reasons why we might study q-exponentiation, but for the purposes of our course, we will mainly use it for generating series of partitiions.

4.3.1 Lemma. *If m is fixed, then*

$$\lim_{N\to\infty} \begin{bmatrix} N \\ m \end{bmatrix}_q = \prod_{i=1}^{m} \frac{1}{1-q^i}.$$

Proof. Recall that $\left[{N \atop m}\right]_q$ is the generating series for partitions with largest part at most m and at most $N-m$ parts, weighted by the size (sum of the parts). Accordingly,

$$\lim_{N\to\infty} \begin{bmatrix} N \\ m \end{bmatrix}_q$$

is just the generating series for partitions with largest part at most m, weighted by size, and is therefore equal to the RHS. □

4.3.2 Theorem. *We have that*

$$\exp_q\left(\frac{t}{1-q}\right) = \prod_{n\geq 0} \frac{1}{1-q^n t}.$$

Proof. Recall that

$$\prod_{i=0}^{n-1} \frac{1}{1-q^i t} = \sum_{j\geq 0} \begin{bmatrix} n+j-1 \\ j \end{bmatrix} t^j.$$

We take the limit as $n \to \infty$ on both sides. On the LHS, we get

$$\prod_{n\geq 0} \frac{1}{1-q^n t},$$

while the RHS becomes,

$$\sum_{j\geq 0} \frac{[n+j-1]!}{[j]![n-1]!} t^j = \sum_{j\geq 0} \frac{1}{[j]!(q-1)^j} t^j.$$

□

One of the most important properties of e^x is that $e^{a+b} = e^a e^b$. We (nearly) have q-analogue of this.

4.3.3 Lemma. *If $BA = qAB$, then $\exp(A+B) = \exp(A)\exp(B)$, for operators A and B.*

We will not prove it, but we might use it.

4.4 Reciprocal

Since

$$[n]_{q^{-1}}! = q^{-\binom{n}{2}}[n]_q!$$

it follows that

$$\exp_{q^{-1}}(t) = \sum_{n\geq 0} \frac{q^{\binom{n}{2}}}{[n]_q!} t^n.$$

This allows us to find the main result here; that something like a reciprocal exsists.

4.4.1 Theorem. *We have* $\exp_q(t)\exp_{q^{-1}}(-t) = 1$.

Proof. Let $A = M_t$ and let $B = -M_t Q$. Then $BA = qAB$ and so $\exp_q(A+B) = \exp_q(A)\exp_q(B)$. Now what is $(A+B)^n(1)$? If $n = 0$, we get 1. If $n > 0$, we get 0. Now we see that $\exp_q(A)(1)\exp_q(B)(1) = 1$.

Since $B^n(1) = (-t)^n q^{\binom{n}{2}}$, we have

$$\exp_q(B)(1) = \sum_{n\geq 0} \frac{q^{\binom{n}{2}}}{[n]_q!} t^n$$

and the result follows. □

4.5 *Durfee square*

The *Durfee square* of a partition π is the largest square in its Ferrer's diagram that contains the top left-hand corner. The Durfee square has side d if d is largest integer such that π has at least d parts of size at least d. See Figure 4.1.

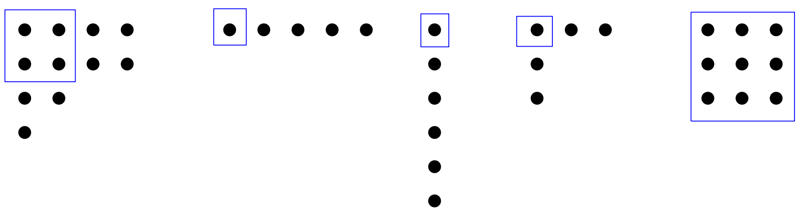

Figure 4.1: Durfee squares of Ferrer's diagrams.

Each partition π with a Durfee square of side d decomposes as:

(a) the Durfee square;

(b) a partition π_1 with at most d parts; and

(c) a partition π_2 with largest part at most d.

Further, if we $|\pi|$ for sum of the parts of π, then

$$|\pi| = d^2 + |\pi_1| + |\pi_2|.$$

Thus the generating series for partitions with Durfee squares of side d (weighted by the sum of the parts) is

$$q^{d^2}\left(\prod_{i=1}^{d}\frac{1}{1-q^i}\right)^2$$

which gives us

$$\sum_{d\geq 0} q^{d^2}\left(\prod_{i=1}^{d}\frac{1}{1-q^i}\right)^2 = \prod_{i\geq 1}\frac{1}{1-q^i}.$$

We used q as a variable but we haven't used the q-bracket, so we must be able to do better than this.

Consider a partition π with at most n parts, each of size at most n and with Durfee square of side d. Then, in the decomposition of π, we have that π_1 has at most d parts, each of size at most $n-d$, and, π_2 has at most $n-d$ parts, each of size at most d. The generating function for partition with at most d parts, each of size at most $n-d$ is

$$\begin{bmatrix} n \\ d \end{bmatrix}.$$

Thus we have that the generating series for such partitions π is

$$q^{d^2}\begin{bmatrix} n \\ d \end{bmatrix}\begin{bmatrix} n \\ n-d \end{bmatrix}.$$

Thus we have

$$\sum_{d=0}^{n} q^{d^2}\begin{bmatrix} n \\ d \end{bmatrix}\begin{bmatrix} n \\ n-d \end{bmatrix} = \begin{bmatrix} 2n \\ n \end{bmatrix},$$

since both sides are equal to the generating function for partitions whose Ferrer's diagram fits into an $n \times n$ box. (This also follows from the q-Vandermonde identity.)

4.6 *Diagonals*

We need another decomposition, with which to count partitions. See Figure 4.2.

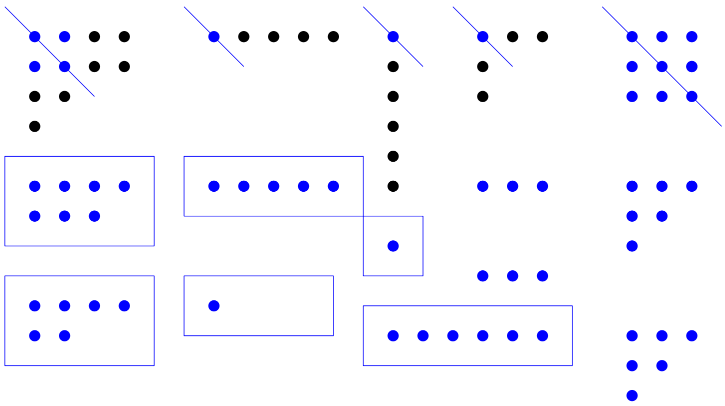

Figure 4.2: Durfee squares of Ferrer's diagrams, divided by diagonals.

If the Durfee square of π has side d, we can view π as the union of two partition, each having exactly d distinct parts, which overlap in the d diagonal elements of the Durfee square.

Let's consider the following

$$A(t) = \prod_{i \geq 1}(1 + q^i t).$$

What is it? This is the generating function where the coefficient t^k is the generating function of partitions with exactly k parts such that all the parts are distinct. The generating function for partitions π with distinct parts, weighted by $|\pi| - k$ (where k is the number of parts) is the coefficient of t^k in

$$\prod_{i \geq 1}(1 + q^{i-1} t) = A(q^{-1} t).$$

From the above decomposition, we conclude that the generating series for all partitions, weighted by the sum of the parts, is the constant term in

$$A(t)A(q^{-1}t^{-1}).$$

In other terms,

$$\prod_{i\geq 1}\frac{1}{1-q^i} = [t^0]\prod_{i\geq 1}(1+q^i t)(1+t^{-1}q^{i-1}).$$

It's not clear (yet) how this could be useful, but what suggests is that the coefficients of t^k could be interesting, even when $k \neq 0$.

4.7 *Jacobi's Triple Product*

The following is a "famous" identity with many applications, one of which is a recurrence for the number of partitions of an integer.

4.7.1 Theorem (Jacobi's Triple Product)**.**

$$\sum_{n=-\infty}^{\infty} q^{\binom{n+1}{2}} t^n = \prod_{n\geq 1}(1-q^n)(1+tq^n)(1+t^{-1}q^{n-1}).$$

4.7.2 Theorem (Jacobi's Triple Product)**.**

$$\sum_{n=-\infty}^{\infty} q^{\binom{n+1}{2}} t^n = \prod_{n\geq 1}(1-q^n)(1+tq^n)(1+t^{-1}q^{n-1}).$$

Substitute q^3 for q and $-q^{-1}$ for t. On the right hand side, we get

$$\prod_{n\geq 1}(1-q^{3n})(1-q^{3n-1})(1-q^{3n-2}) = \prod_{n\geq 1}(1-q^n).$$

This is some series that Euler studied. He multiplied out the first couple of products of this and he observed the coefficients are always $0, 1, -1$ and he formulated a guess for when each values occurs. We can already prove Euler's guess, using the Jacobi's Triple Product Theorem. Hopefully this motivates why we want to prove such a statement.

Proof. [Jacobi's Triple Product]

Let $F(t)$ be defined by

$$F(t) = \prod_{n\geq 1}(1+tq^n)(1+t^{-1}q^{n-1}).$$

Then,

$$\begin{aligned} F(qt) &= \prod_{n\geq 1}(1+tq^{n+1})(1+t^{-1}q^{n-2}) \\ &= \frac{1+t^{-1}q^{-1}}{1+tq}\prod_{n\geq 1}(1+tq^n)(1+t^{-1}q^{n-1}) \\ &= q^{-1}t^{-1}F(t). \end{aligned}$$

Let f_n be the coefficients of t^n in $F(t)$; that is

$$F(t) = \sum_{n=-\infty}^{\infty} f_n(q)t^n.$$

Then

$$\sum_{n=-\infty}^{\infty} f_n(q)q^n t^n = F(qt) = q^{-1}t^{-1}F(t) = q^{-1}t^{-1} \sum_{n=-\infty}^{\infty} f_n(q)t^n$$

whence

$$f_n(q) = q^n f_{n-1}(q).$$

We can easily solve this recurrence and obtain

$$f_n(q) = q^{\binom{n+1}{2}} f_0(q).$$

Since $f_0(q)$ is the constant term in $F(t)$, which we found in the last class, we obtain that

$$F(t) = \left(\prod_{n\geq 0}(1-q^n)^{-1}\right) \sum_{n=-\infty}^{\infty} q^{\binom{n+1}{2}} t^n$$

and rearranging this gives the desired result. □

4.8 *A second proof (with q's)*

We will rederive the triple product identity. Let

$$A = M_t Q, \quad B = Q.$$

where $M_t : C(t) \to tC(t)$ and $Q : C(t) \to C(qt)$. Again $BA = qAB$ and so

$$(A+B)^{m+n} = \sum_{k=0}^{m+n} \begin{bmatrix} m+n \\ k \end{bmatrix} A^k B^{m+n-k}.$$

Thus

$$A^{-m}(A+B)^{m+n} = \sum_{k=0}^{m+n} \begin{bmatrix} m+n \\ k \end{bmatrix} A^{k-m} B^{m+n-k} = \sum_{\ell=-m}^{n} \begin{bmatrix} m+n \\ m+\ell \end{bmatrix} A^{\ell} B^{n-\ell}.$$

Now applying this to 1, we get

$$(A+B)(1) = t+1, \quad (A+B)^2(1) = (A+B)(t+1) = (t+1)(qt+1)$$

eventually, we find

$$A^{-m}(A+B)^{m+n}(1) = (1+q^m t^{-1})(1+q^{m-1}t^{-1})\cdots(1+qt^{-1})(1+t)(1+qt)\cdots(1+q^n t)$$

and on the other side we get

$$\sum_{\ell=-m}^{n} \begin{bmatrix} m+n \\ m+\ell \end{bmatrix} A^{\ell} B^{n-\ell}(1) = \sum_{\ell=-m}^{n} \begin{bmatrix} m+n \\ m+\ell \end{bmatrix} q^{\binom{\ell}{2}} t^{\ell}.$$

Now we take $m = n$. Then

$$\sum_{\ell=-n}^{n} \begin{bmatrix} 2n \\ n+\ell \end{bmatrix} q^{\binom{\ell}{2}} t^{\ell}$$

and

$$\lim_{n\to\infty} \begin{bmatrix} 2n \\ n+\ell \end{bmatrix} = \prod_{i\geq 1} \frac{1}{1-q^i}.$$

We have that

$$\begin{bmatrix} 2n \\ n+\ell \end{bmatrix} = \frac{[2n][2n-1]\cdots[n+\ell+1]}{[n-\ell]!} = \frac{(q^{2n}-1)\cdots(q^{n+\ell+1}-1)}{(q^{n-\ell}-1)\cdots(q-1)}$$

[This was not the right way, sorry, don't do this.] The actual way to see this is, is to observe that the RHS is the generating for all partitions weighted by the sum of the parts and,

$$\begin{bmatrix} 2n \\ n+\ell \end{bmatrix}$$

is the generating series for all partitions whose Ferrer's diagrams fit in a $n+\ell \times n-\ell$ box. When we take the limit, we get all partitions. Now, we get

$$\lim_{n\to\infty} \sum_{\ell=-n}^{n} \begin{bmatrix} 2n \\ n+\ell \end{bmatrix} q^{\binom{\ell}{2}} t^{\ell} = \prod_{i\geq 1} \frac{1}{1-q^i} \sum_{\ell=-\infty}^{\infty} q^{\binom{\ell}{2}} t^{\ell}$$

while on the LHS, we

$$\prod_{n\geq 1} (1+tq^n)(1+t^{-1}q^{n-1})$$

and the result follows.

4.9 *Euler's Pentagonal Number Theorem*

The numbers $m(3m-1)/2$ are known as the *pentagonal numbers*. See Figure 4.3.

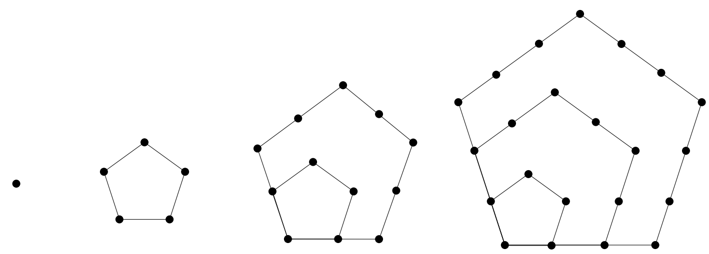

Figure 4.3: Euler pentagonal numbers.

If $\mathcal{S}$ is a set of partitions, let $p(\mathcal{S}, n)$ denote the number of partitions of n in $\mathcal{S}$. Let $\mathcal{D}$ denote the set of all partitions with distinct parts. Let $\mathcal{E}$ denote the set of the partitions with an even number of parts and let $\mathcal{O}$ denote the number of partitions with an odd number of parts.

4.9.1 Theorem.

$$p(\mathcal{D}\cap\mathcal{E}, n) - p(\mathcal{D}\cap\mathcal{O}, n) = \begin{cases} (-1)^m, & n = \frac{m(3m\pm 1)}{2}; \\ 0, & \textit{otherwise}. \end{cases}$$

Proof. We want to "pair" the partitions of n with distinct parts. See Figure 4.4. Let $a_1, \ldots, a_k$ be a partition of n where $a_i > a_{i+1}$ for $i = 1, \ldots, k-1$. Let j be the smallest integer such $a_{j+1} < a_j - 1$; if $a_1 > a_2 + 1$, then we say that $j = 1$, and if $a_{i+1} = a_i - 1$ for

Figure 4.4: We move the smallest part to the right-most diagonal, or move the right-most diagonal to form a new (smallest) part.

$i = 1, \dots, k-1$, then we say that $j = k$. (Intuitively, we want to either put the smallest part on the right-most diagonal, or we want to remove the right-most diagonal and put as the last part.)

If $j < a_k$, then we pair $a_1, \dots, a_k$ with $a_1 - 1, \dots, a_j - 1, a_j, \dots, a_k, j$. If $j \geq a_k$, we can remove the last part a_k and add to the rightmost diagonal. Exercise: verify that this gives a bijection (amongst the partitions for which it is defined) which changes the parity of the number of parts.

When does this fail? This is not defined when $j = a_k = k$ or $j = k = a_k - 1$. If $j = a_k = k$, what is n? We get

$$n = j + (j+1) + \cdots + (2j-1) = \frac{3j^2 - j}{2}.$$

If $j = k = a_k - 1$, we get

$$n = (j+1) + \cdots + (2j) = \frac{3j^2 + j}{2}.$$

In each case, we need only verify (left as exercise) that there is a single partition with j parts for which our bijection is not defined, and find the parity of its number of parts. □

5 Solving recurrences

Recall Problem 4 from Assignment 2. Let $S = \{1, 2\}$ and $C_n(S)$ is the set of sequences with elements in S, where the sum of the terms is n. Let $c_n = |C_n(S)|$ and we see that $c_0 = 1$, $c_1 = 1$, $c_2 = 2$ and in general

$$c_{n+1} = c_n + c_{n-1}.$$

How would we find the general term for c_n? Or the generating function?

Let $G(x)$ be the generating function; that is

$$G(x) = \sum_{n=0}^{\infty} c_n x^n.$$

We have

$$c_n = c_{n-1} + c_{n-2}.$$

for $n \geq 2$. We multiply by x^n on both sides to get

$$c_n x^n = c_{n-1} x^n + c_{n-2} x^n$$

and then we take the sum for $n = 2, 3, \ldots$ to get

$$\sum_{n=2}^{\infty} c_n x^n = \sum_{n=2}^{\infty} c_{n-1} x^n + \sum_{n=2}^{\infty} c_{n-2} x^n.$$

We can rewrite this in terms of the generating function as follows:

$$G(x) - c_1 x - c_0 = x(G(x) - c_0) + x^2 G(x).$$

We can recall that $c_0 = c_1 = 1$ and then

$$\begin{aligned} G(x) - x - 1 &= xG(x) - x + x^2 G(x) \\ G(x)(1 - x - x^2) &= 1 \\ G(x) &= \frac{1}{1 - x - x^2} \end{aligned}$$

and we have obtained $G(x)$ as a rational function.

5.1 Homogeneous recurrence equations

In general, suppose that we have the recurrence for the sequence $c_0, c_1, \ldots$

$$c_n + q_1 c_{n-1} + \cdots + q_k c_{n-k} = 0, \quad n \geq k \tag{5.1.1}$$

where $q_1, \dots, q_k$ are constants where $q_k \neq 0$. To uniquely determine the sequence $c_1, c_2, \dots$, we need to specify $c_0, c_1, \dots, c_{k-1}$ as the initial conditions. We want to express c_n as a function of n. This is a *homogeneous equation* since the RHS is 0.

First, we define the polynomial

$$E(x) = x^k + q_1 x^{k-1} + \cdots + q_k$$

to be the *characteristic polynomial* of the recurrence.

5.1.1 Theorem. *Suppose $\{c_n\}$ satisfies (5.1.1). If the roots of the characteristic polynomial are β_i with multiplicity m_i for $i = 1, \dots, j$, then*

$$\begin{aligned} c_n = {} & (b_{11} + b_{12} n + \cdots + b_{1m_1} n^{m_1 - 1}) \beta_1^n + \cdots \\ & + (b_{j1} + b_{j2} n + \cdots + b_{jm_j} n^{m_j - 1}) \beta_j^n \end{aligned}$$

where the b_{ij} are constants that depend on the initial conditions.

Proof. Multiply (5.1.1) by x^n and sum over all $n \geq k$ to get

$$\sum_{n=k}^{\infty} c_n x^n + \sum_{n=k}^{\infty} q_1 c_{n-1} x^n + \cdots + \sum_{n=k}^{\infty} q_k c_{n-k} x^n = 0.$$

Let $C(x) = \sum_{n=0}^{\infty} c_n x^n$ and $C_m(x) = \sum_{n=0}^{m} c_n x^n$. So,

$$\sum_{n=k}^{\infty} c_{n-i} x^n = \sum_{t=k-i}^{\infty} c_t x^{t+i} = x^i \sum_{t=k-i}^{\infty} c_t x^t = x^i (C(x) - C_{k-i-1}(x))$$

with $C_{-1}(x) = 0$ by convention. Now we can substitute all of these into our summation and obtain an expression with $C(x)$ and $C_m(x)$ from which we get our rational function.

Now we get

$$C(x) = \frac{P(x)}{Q(x)}$$

where

$$Q(x) = 1 + q_1 x + \cdots + q_k x^k$$

and

$$P(x) = C_{k-1}(x) + q_1 x C_{k-2}(x) + \cdots + q_{k-1} x^{k-1} C_0(x).$$

What is the degree of $P(x)$? It is at most $k - 1$. We can continue and see

$$Q(x) = x^k (x^{-k} + q_1 x^{1-k} + \cdots + q_k) = x^k E(x^{-1}).$$

But we are given that

$$E(x) = (x - \beta_1)^{m_1} (x - \beta_2)^{m_2} \cdots (x - \beta_j)^{m_j}.$$

Since $P(x)$ has degree at most $k - 1$, we can break down $C(x)$ using a partial fraction expansion as follows:

$$C(x) = \frac{A_{1,1}}{1 - \beta_1 x} + \cdots + \frac{A_{1,m_1}}{(1 - \beta_1 x)^{m_1}} + \cdots + \frac{A_{j,1}}{1 - \beta_j x} + \cdots + \frac{A_{j,m_j}}{(1 - \beta_j x)^{m_j}}$$

where $A_{i,\ell}$ are constants.

But $c_n = [x^n]C(x)$ and so we have

$$c_n = \left(A_{1,1} + A_{1,2}\binom{n+1}{1} + \cdots + A_{1,m_1}\binom{n+m_1-1}{m_1-1}\right)\beta_1^n + \cdots$$
$$+ \left(A_{j,1} + A_{j,2}\binom{n+1}{1} + \cdots + A_{j,m_j}\binom{n+m_j-1}{m_j-1}\right)\beta_j^n$$

by using the Binomial Theorem for negative integer powers. (For example, look at $\frac{1}{(x-\beta_1)^{m_1}} = (x-\beta_1)^{-m_1}$ and we get the coefficient of x^n from the Binomial Theorem.)

Note that we can determine the $A_{i,\ell}$s by using the equations that we get for $c_0, \ldots, c_{k-1}$. □

In practice, the theorem just says that this is possible and the most important content is in the proof.

Example: Find c_n explicitly, where

$$c_n - 4c_{n-1} + 5c_{n-2} - 2c_{n-3} = 0, \quad n \geq 3$$

with initial conditions $c_0 = 1, c_1 = 1$ and $c_2 = 2$

Solution. The characteristic polynomial[1] is as follows:

$$x^3 - 4x^2 + 5x - 2 = (x-1)^2(x-2),$$

which has roots $1, 2$ with multiplicities $2, 1$, respectively. Thus we see that

$$C(x) = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1-2x}$$

for some A, B, C. Thus

$$c_n = [x^n]C(x) = A + Bn + C(2)^n$$

for all n. Then, in particular, we obtain from the initial conditions

$$1 = A + 0 + C$$

$$1 = A + B + 2C$$

$$2 = A + 2B + 4C$$

which you can solve to get $A = 0$, $B = -1$ and $C = 1$. Thus we obtain that $c_n = 2^n - n$.

From the theorem, we can see that these sequence will tend to grow exponentially. It is somewhat intuitive that exponential sequences will satisfy such recurrence.

For example, if $c_n = (3^n - 2^n)^2$ then

$$c_n = 9^n - 2 \cdot 6^n + 4^n.$$

We can now read off the recurrence relation from the proof of the theorem: the characteristic polynomial could be

$$(x-9)(x-6)(x-4) = x^3 - 19x^2 + 114x - 216$$

thus

$$c_n - 19c_{n-1} + 114c_{n-2} - 216c_{n-3} = 0, \quad n \geq 3.$$

Now, one can verify that this is the correct recurrence relation, by substituting the explicit expressions for c_n, c_{n-1} and so on.

[1] Note that this is the characteristic polynomial of the matrix

$$A = \begin{pmatrix} 4 & -5 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and that

$$A \begin{pmatrix} c_{n-1} \\ c_{n-2} \\ c_{n-3} \end{pmatrix} = \begin{pmatrix} c_n \\ c_{n-1} \\ c_{n-2} \end{pmatrix}.$$

5.2 Nonhomogeneous recurrence equations

Suppose $b_0, b_1, \ldots$ satisfies the recurrence

$$b_n + q_1 b_{n-1} + \cdots + q_k b_{n-k} = f(n), \quad n \geq k \tag{5.2.1}$$

where $q_1, \ldots, q_k$ are constants and $f(n)$ is a function in n. For the sequence to be uniquely define, we also need initial conditions for $b_0, \ldots, b_{k-1}$.

5.2.1 Theorem. *Suppose $a_0, a_1, \ldots$ is any solution to (5.2.1). Then the general solution is*

$$b_n = a_n + c_n$$

where c_n is given by the previous theorem and the initial conditions for c_i are determined by those for b_i.

Proof. We are given that

$$a_n + q_1 a_{n-1} + \cdots + q_k a_{n-k} = f(n)$$

for all $n \geq k$. Subtracting this equation from (5.2.1), we get

$$(b_n - a_n) + q_1(b_{n-1} - a_{n-1}) + \cdots + q_k(b_{n-k} - a_{n-k}) = 0$$

Now $b_n - a_n$ satisfies the homogeneous version of the recurrence. □

Problem: Solve

$$b_n - 4b_{n-1} + 5b_{n-2} - 2b_{n-3} = 24(-1)^n, \quad n \geq 3$$

with $b_0 = -1$, $b_1 = -3$ and $b_2 = 2$.

Solution. We have to guess. Let's guess that $a_n = \alpha(-1)^n$ for some constant α. Now we substitute our guess into the recurrence and get

$$\alpha(-1)^n - 4\alpha(-1)^{n-1} + 5\alpha(-1)^{n-2} - 2\alpha(-1)^{n-3} = \alpha(-1)^n(1+4+5+2) = 12\alpha(-1)^n.$$

If we take $\alpha = 2$, this is some sequence that satisfies the recurrence relation. Now our theorem tells us that the general solution for b_n is

$$b_n = 2(-1)^n + A + Bn + C2^n, \quad n \geq 0.$$

We now solve for A, B, C using the initial conditions; [INSERT ON YOUR OWN] and also solve. In the end, you should get

$$b_n = 2(-1)^n - 2 + 3n - 2^n.$$

In general, this could be difficult. The key is try a_n to be multiples of the RHS. For example if the RHS is $2n - 1$, guess $a_n = \alpha n + \beta$, substitute into the recurrence and solve for α, β.

5.3 Asymptotics

We say that that c_n is *asymptotic* to $g(n)$ as $n \to \infty$, and we write $c_n \sim g(n)$ if

$$\lim_{n\to\infty} \frac{c_n}{g(n)} = 1.$$

For example, if we have a recurrence relation where the roots of the characteristic polynomial are $\beta_1, \ldots, \beta_m$, where β_1 is the unique largest in absolute value, we can say something about the asymptotics.

In the general expression for c_n, the coefficient of β_1^n is some polynomial in n. Let b_1 be the coefficient in from of the largest power of n. Then, one can show

$$\lim_{n\to\infty} \frac{c_n}{b_1 n^{m_1} \beta_1^n} = 1,$$

since $\frac{\beta_i}{\beta_1}$ is less than 1 in absolute value. We write $c_n \sim b_1 n^{m_1} \beta_1^n$ and this answers the question "roughly how big does c_n get for large n?". For example, $b_n \sim -2^n$ for b_n in the last example.

5.4 *More Recurrences*

Let $F(t) = \prod_{i\geq 0}(1-tq^i)^{-1} = \sum_{i\geq 0} c_i t^i$. Recall that $F(t)$, consider in variables t, q is the generating function for all partitions weighted by the number of parts and the sum of the parts.

We have previously shown that $F(qt) = (1-t)F(t)$. But now

$$F(qt) = \sum_{i\geq 0} c_i q^i t^i = (1-t)\left(\sum_{i\geq 0} c_i t^i\right)$$

and we extract the coefficient of t^n from both sides to get

$$c_n q^n = c_n - c_{n-1}$$

and thus $c_n = (1-q^n)^{-1} c_{n-1}$ for $n \geq 1$ and $c_0 = 1$. Thus

$$c_n = \prod_{j=1}^{n} (1-q^j)^{-1}.$$

What is this? This is the generating function for partition with largest part at most n, weighted by the sum of the parts. This method for extracting the coefficient of one of the variables in called Euler's device.

Using this device, you can also show

$$\prod_{i\geq 0}^{n} (1-tq^i)^{-1} = 1 + \sum_{k\geq 1} t^k \prod_{i=1}^{k} \frac{1-q^{n+i}}{1-q^i}.$$

Here the LHS is the generating for partitions with largest part at most n, weighted by the sum of the parts and the number of parts. The product in the sum on the RHS is the generating function for partitions with at most k parts and largest part at most n.

5.5 *Rogers and Ramanujan*

A partition is 2-distinct if any two parts differ by at least 2. First, we find the generating series for 2-distinct partitions.

Suppose $n_1, \ldots, n_k$ is an increasing sequence where terms are 2-distinct and sums to n. Then the sequence

$$n_1 - 1, n_2 - 3, \cdots, n_k - (2k-1)$$

is a non-decreasing sequence and sums to $n - k^2$. This gives a bijection between 2-distinct partitions of n with k parts and partitions of $n - k^2$ with at most k parts. (Something to check here)

The generating function for partitions with at most k parts is

$$\prod_{i=1}^{k} \frac{1}{1-q^i}$$

and consequently the generating series of 2-distinct partitions with exactly k parts is

$$q^{k^2} \prod_{i=1}^{k} \frac{1}{1-q^i}.$$

Now we can conclude that the generating series of 2-distinct partition is

$$\sum_{k \geq 0} \frac{q^{k^2}}{(1-q)(1-q^2)\cdots(1-q^k)}.$$

5.5.1 Theorem.

$$\sum_{k \geq 0} \frac{q^{k^2}}{(1-q)(1-q^2)\cdots(1-q^k)} = \prod_{i \geq 1} \frac{1}{(1-q^{5i-4})(1-q^{5i-1})}$$

$$\sum_{k \geq 0} \frac{q^{k^2+k}}{(1-q)(1-q^2)\cdots(1-q^k)} = \prod_{i \geq 1} \frac{1}{(1-q^{5i-3})(1-q^{5i-2})}$$

Proof. Use Jacobi's triple product (choose appropriate substitutions for q, t). □

Note that the second series is the generating series for 2-distinct partitions with smallest part at least 2.

What does this mean? The RHS is

$$\prod_{i \equiv 1,4 \mod 5} \frac{1}{1-q^i}$$

is the generating function for partitions where every part has size $1, 4 \mod 5$. Can you think of a bijection for this?

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