Using SageMath in Research

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Université libre de Bruxelles

CMS mini-course 2019, Regina

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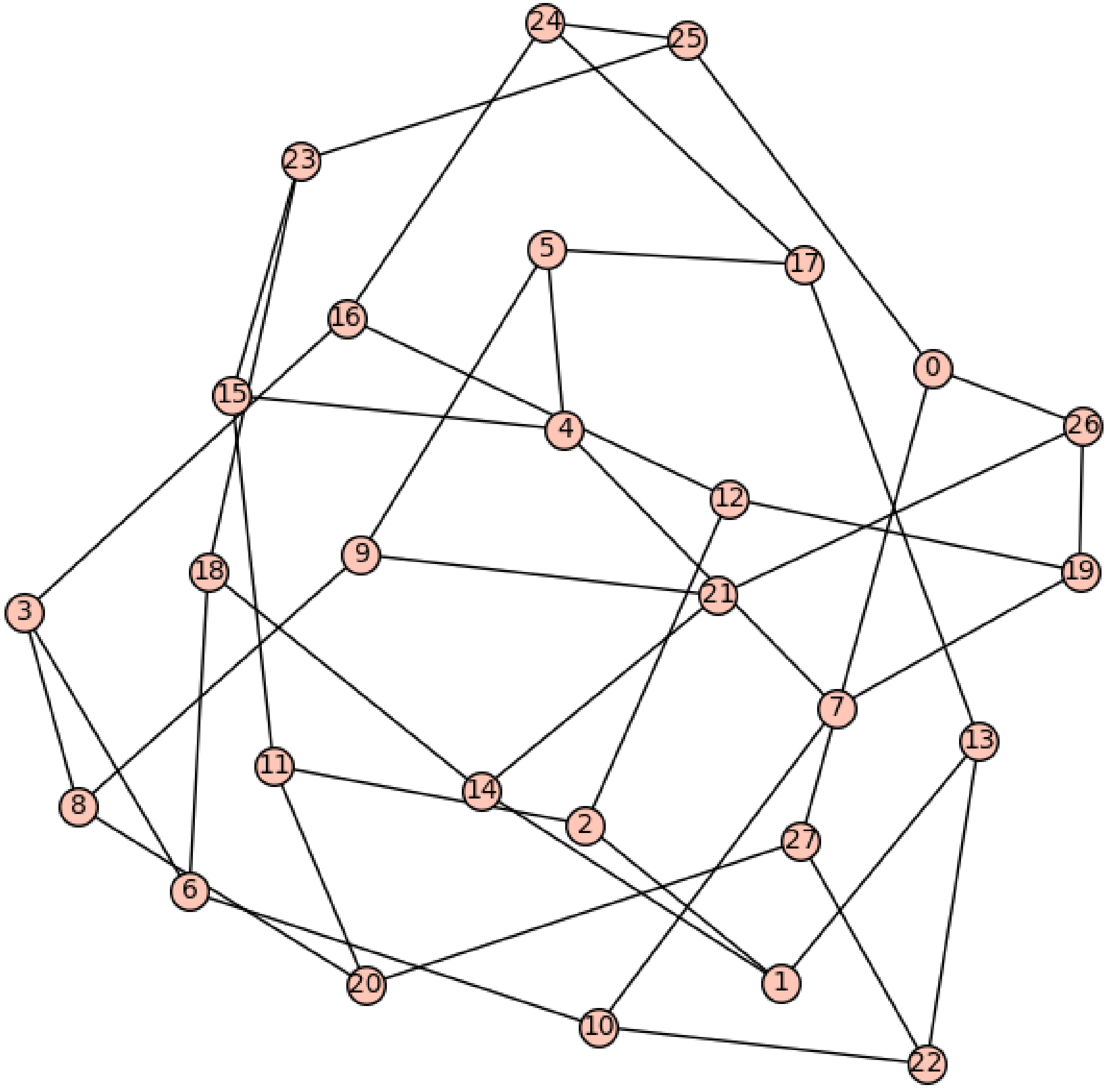
SageMath was started by William Stein.

Guess this graph



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- To introduce everyone to my favourite graph

The When/Where/Why

Checking conjectures and counterexamples

Theorem (Mohar 2013)

> Every connected subcubic bipartite graph that is not isomorphic to the Heawood graph has at least one (in fact a positive proportion) of its eigenvalues in the interval $[-1, 1]$.

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We construct an infinite family of connected cubic bipartite graphs which have no eigenvalues in the open interval $(-1, 1)$.

Found by searching the census of cubic vertex-transitive graphs.

Visualizing new mathematical objects

Continous quantum walk

Transition matrix of the quantum walk

$$U(t) = e^{itA}$$

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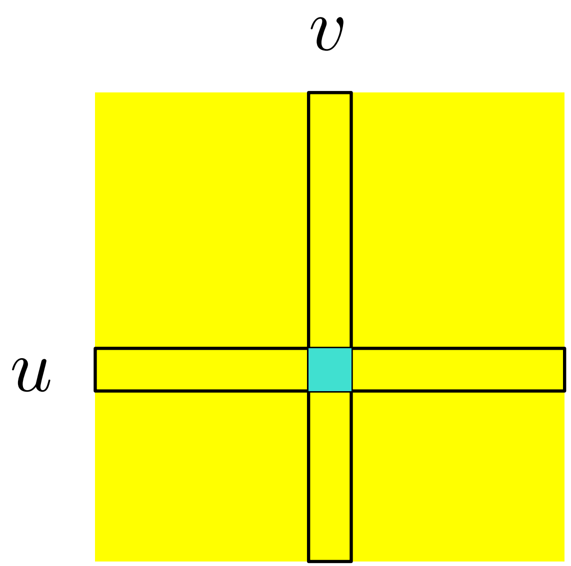

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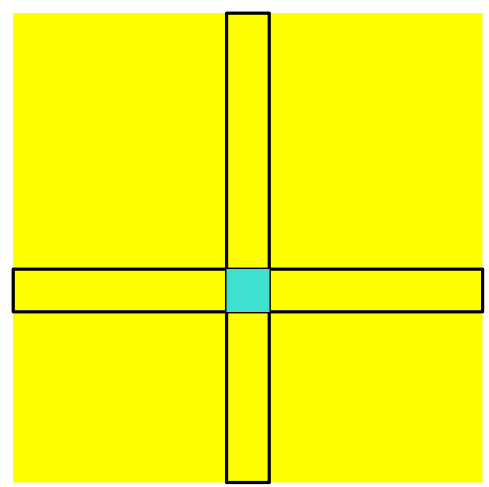
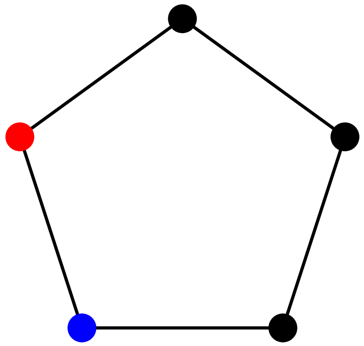


$t \mapsto (x_t, y_t)$

$U(t)_{u,v} = x_t + iy_t$

Plotting

$U(t)_{a,b}$ for $t \in [0, 100]$.



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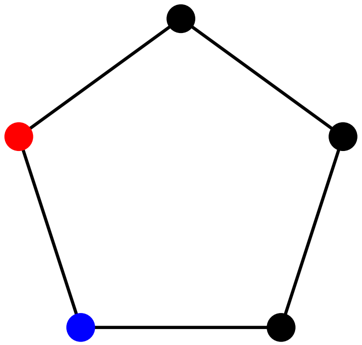
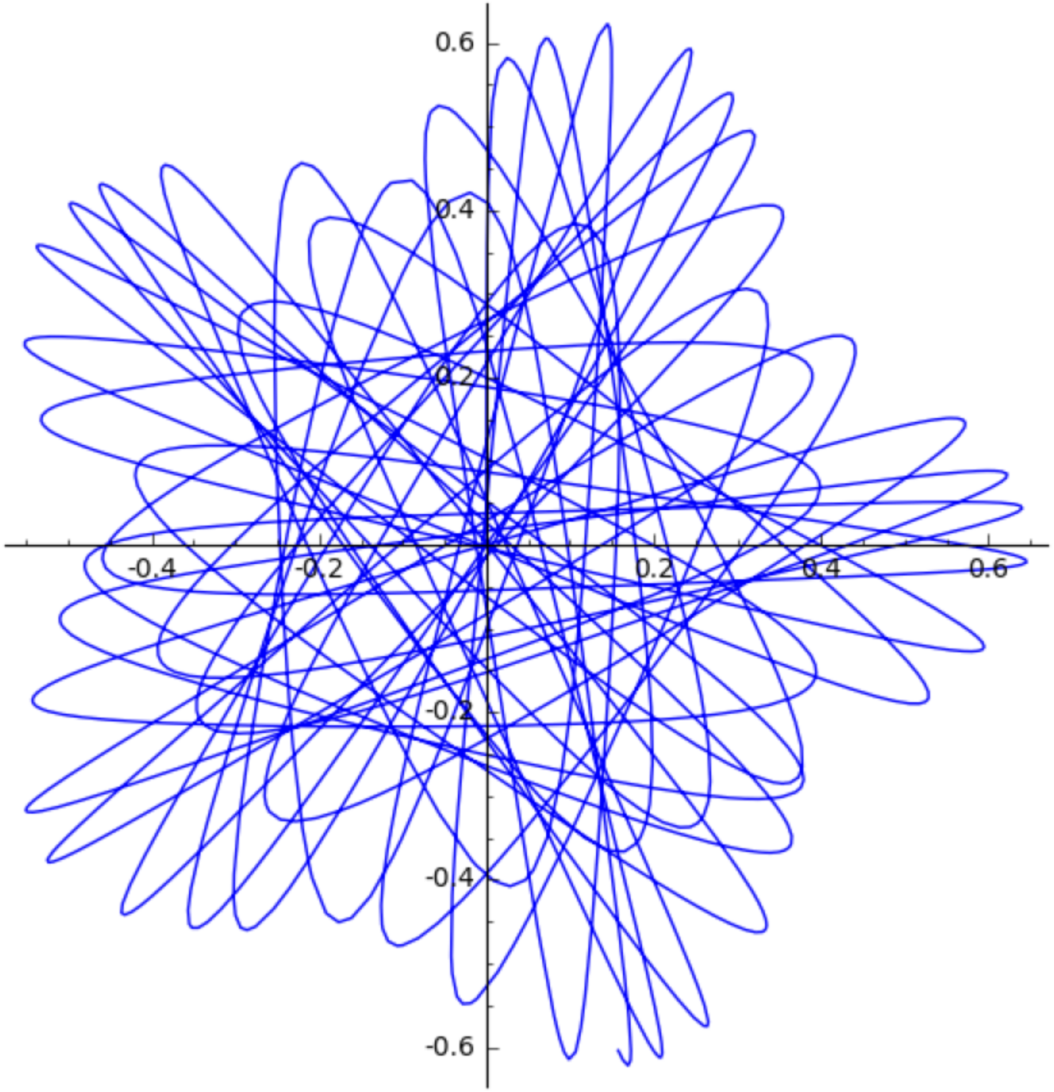
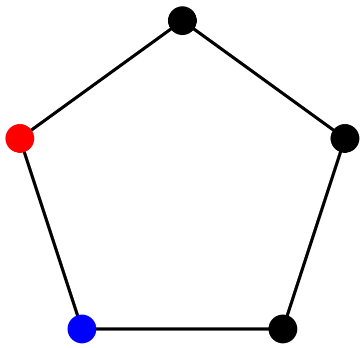




Plotting

$U(t)_{a,b}$ for $t \in [0, 500]$.



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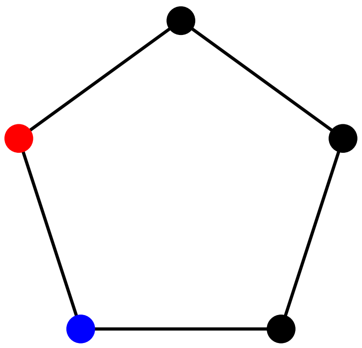

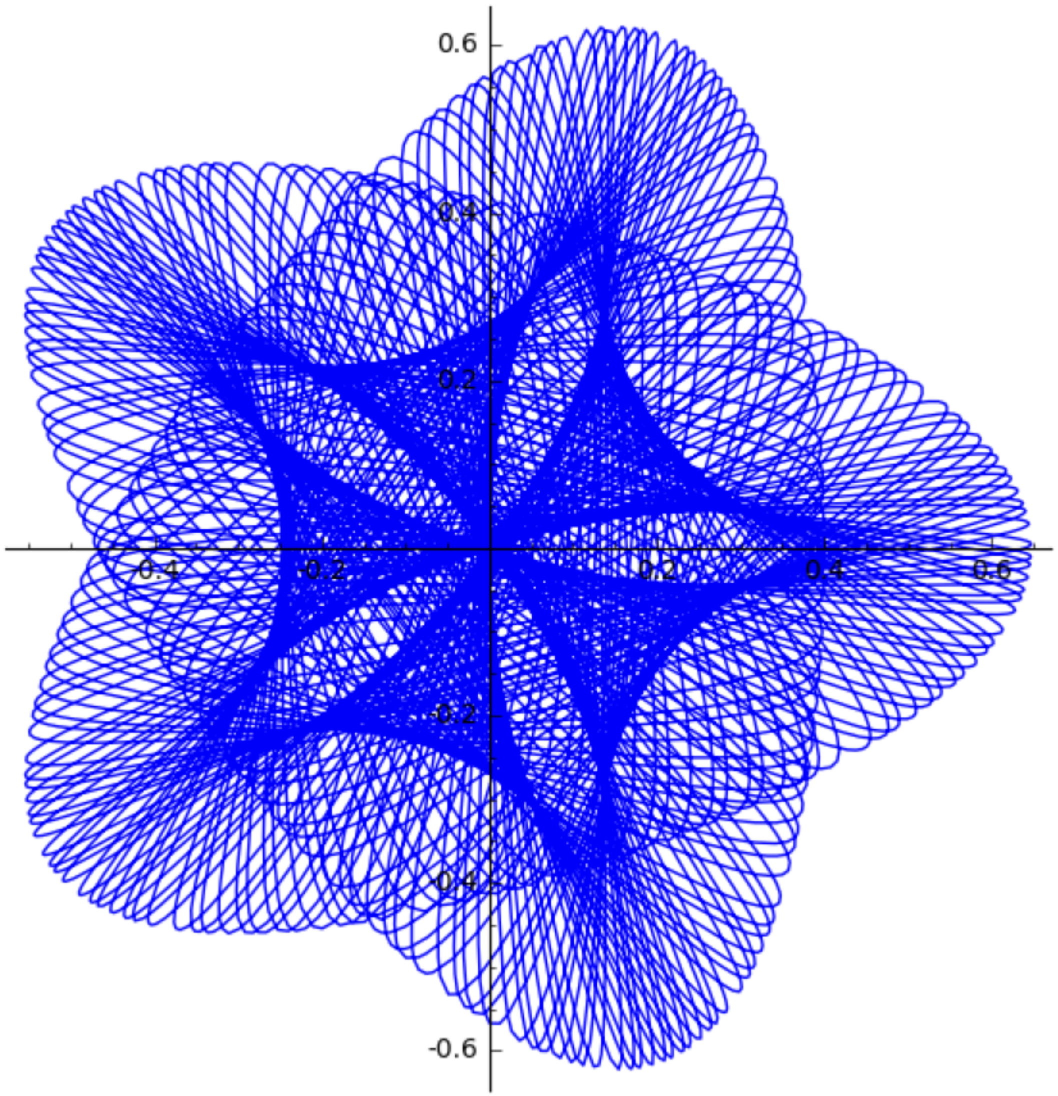

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- .... many other uses!

Get to know SageMath!

sage: runfile cms-guessthisgraph.sage

g1, g2, g3 have been defined

sage: g1.

```
clique_number()        is_regular()
degree()               is_bipartite()
diameter()             is_clique()
girth()                is_chordal()
complement()           is_planar()
                       is_connected()
```

Constructing an awesome graph

We will demonstrate SageMath by constructing a specific graph. We will use the following ingredients:

Projective geometry, hyperovals, finite fields.

Strongly regular graphs

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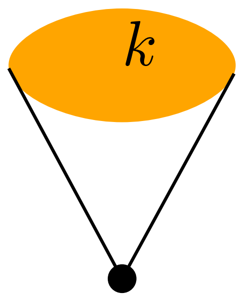

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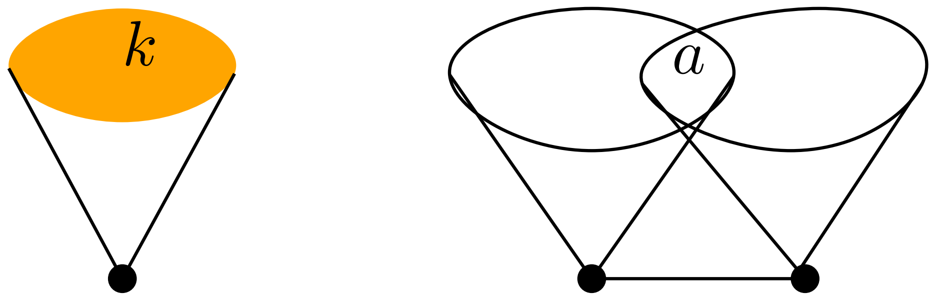

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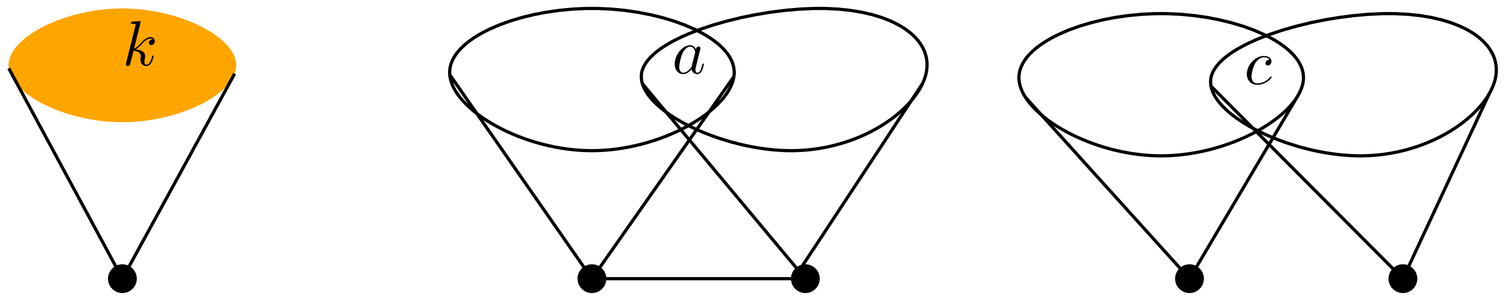

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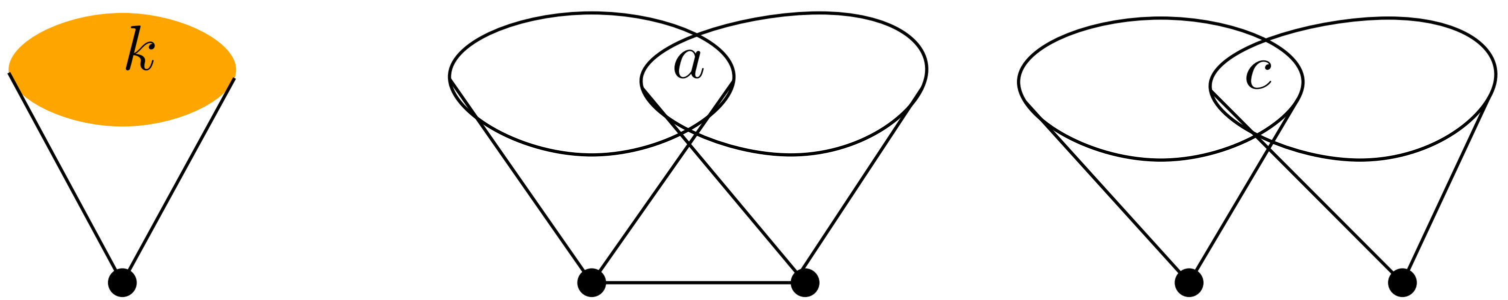

Example: $(25, 8, 3, 2)$.

How do we construct strongly regular graphs?

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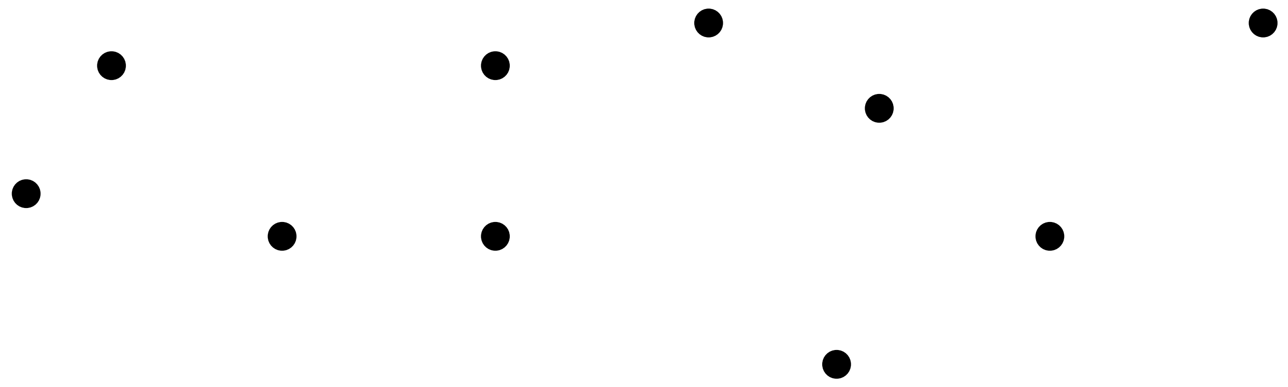

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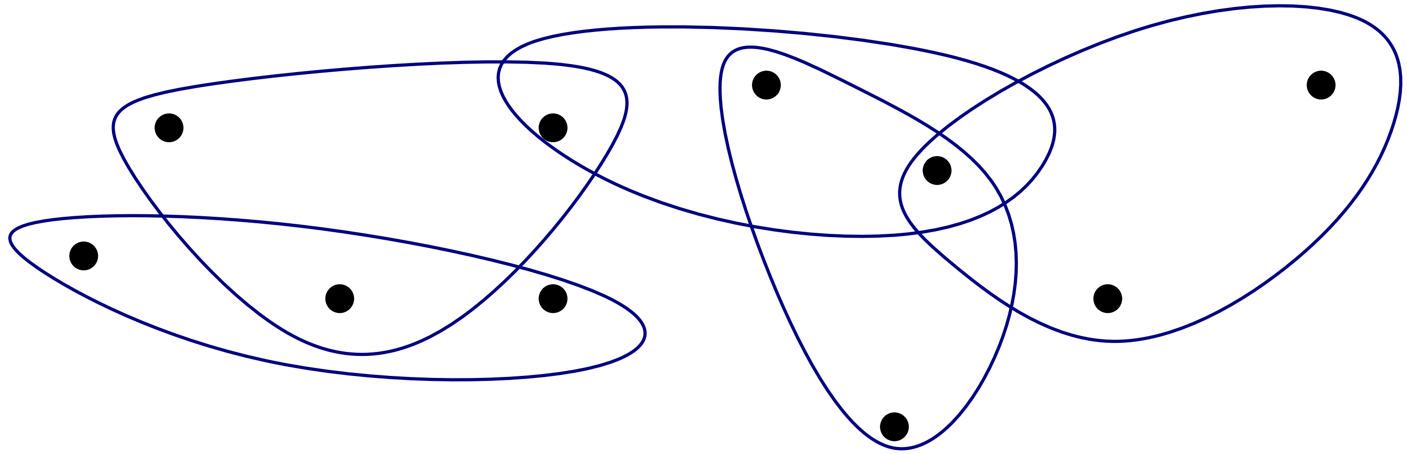

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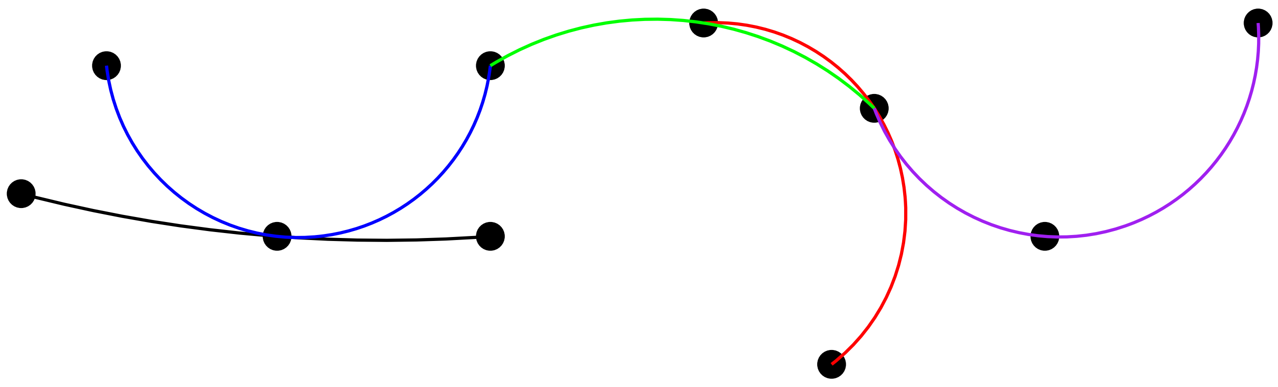

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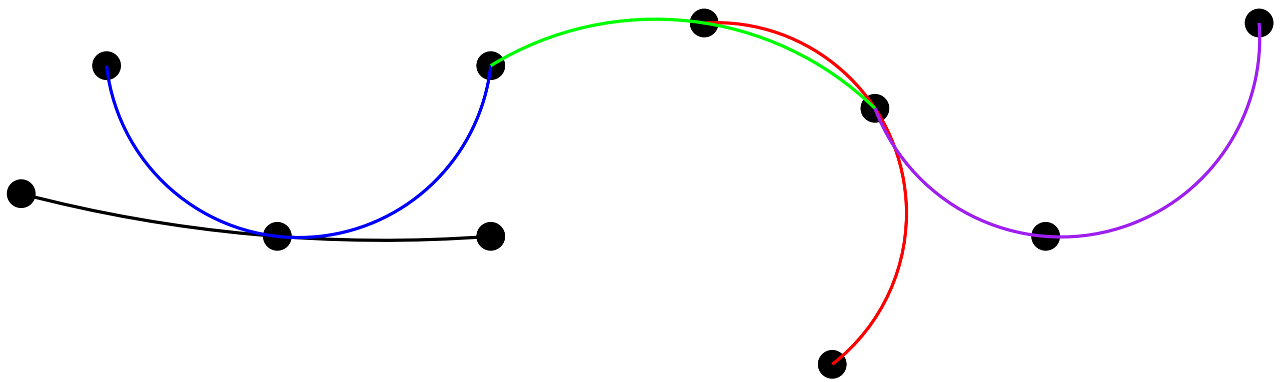

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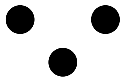
vertices: 

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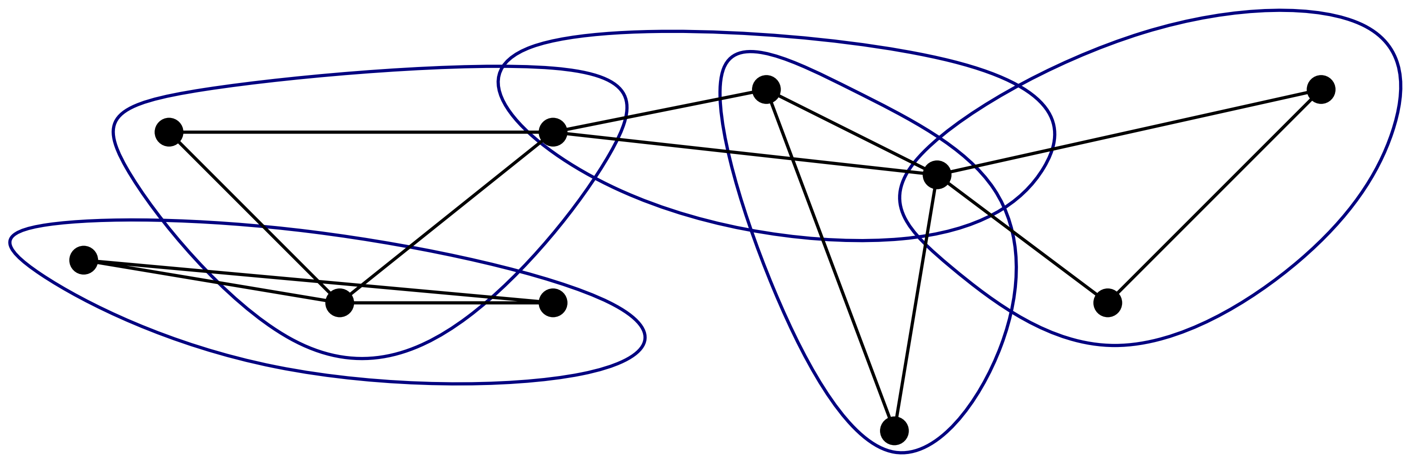

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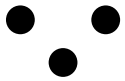
vertices: 

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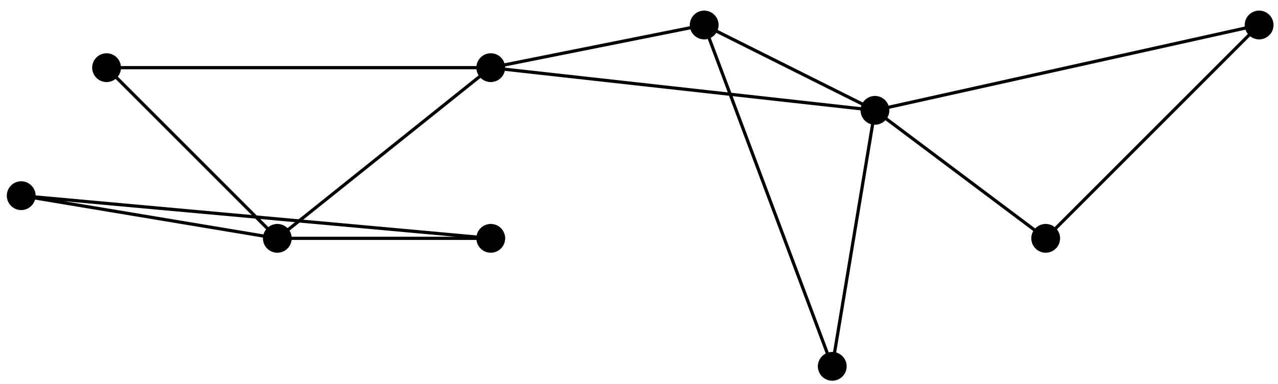

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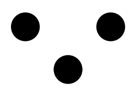
vertices: 

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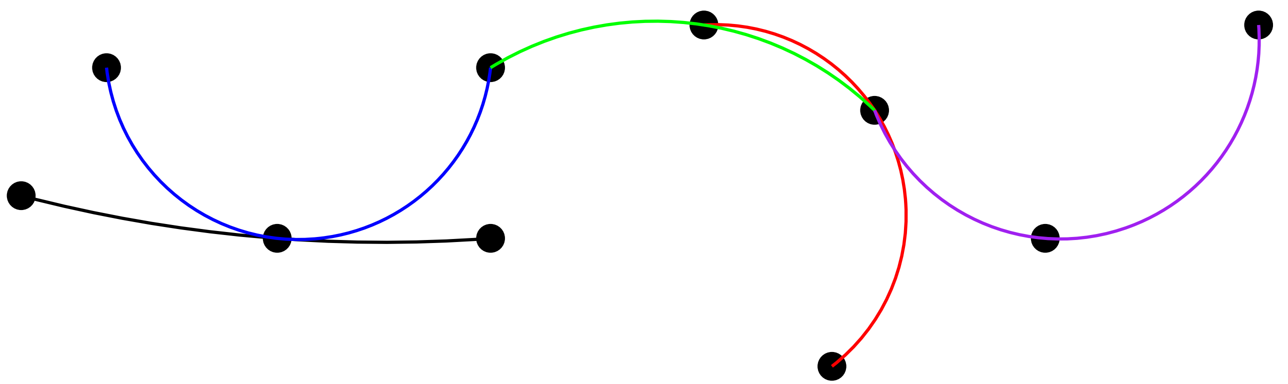

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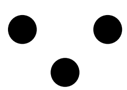
vertices: 

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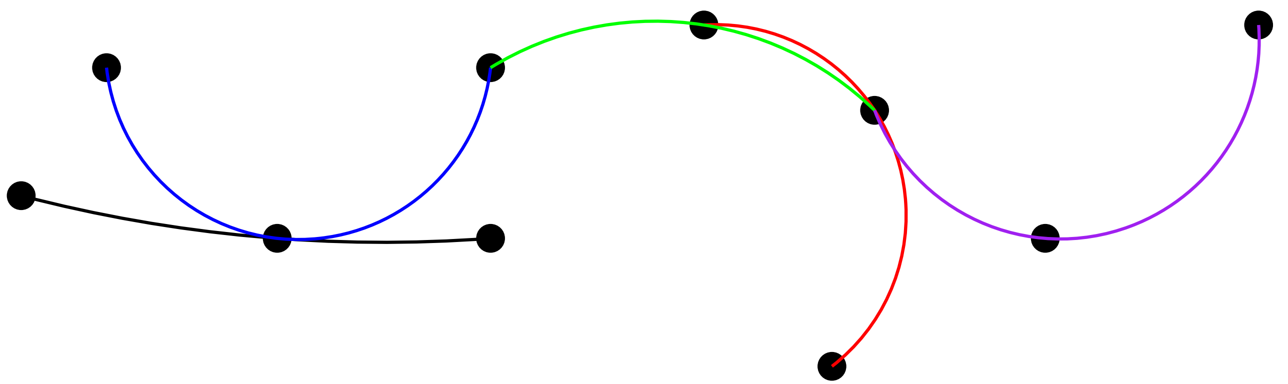

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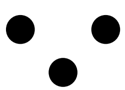
vertices: 

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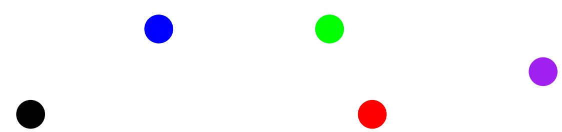

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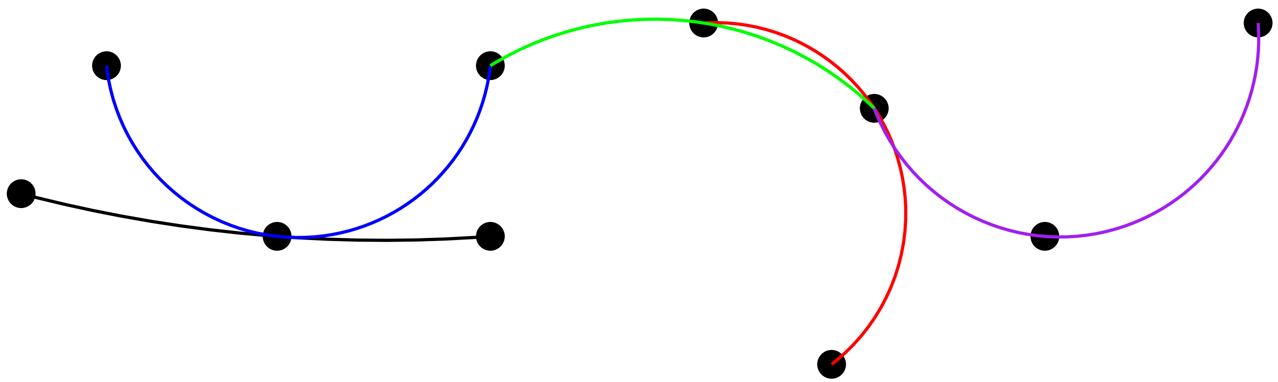

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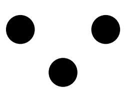
vertices: 

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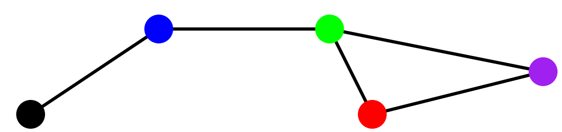

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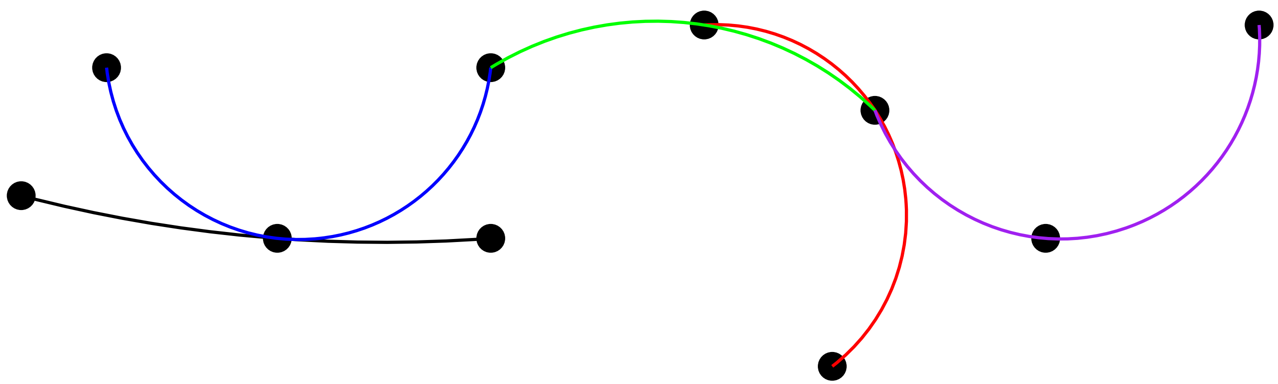

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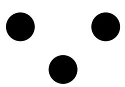
vertices: 

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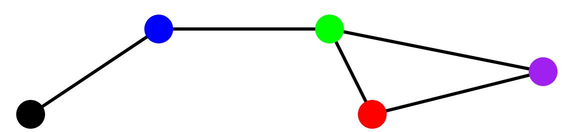

We get a strongly regular graph as a point (or block) graph if we pick a "nice" incidence structure.

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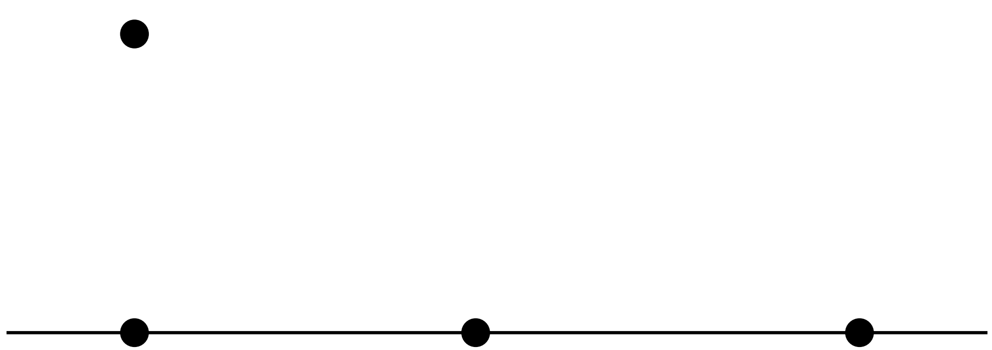

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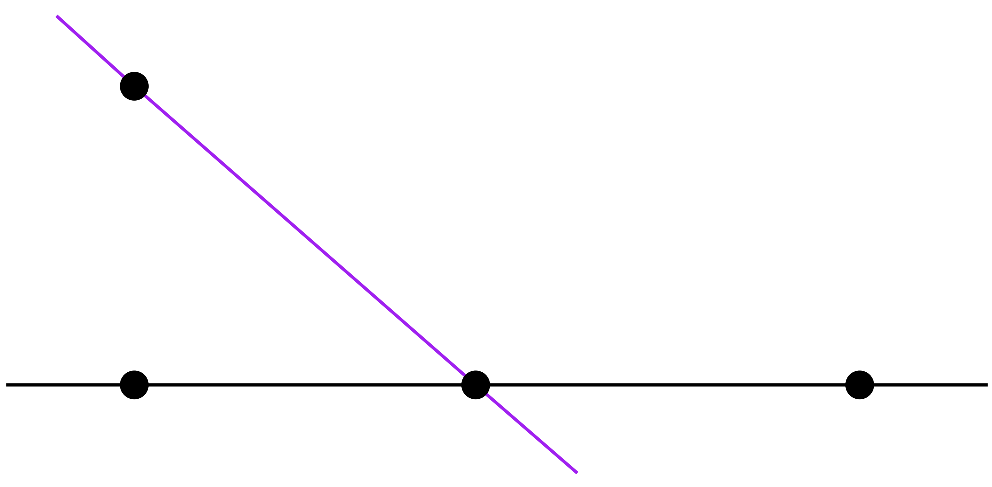

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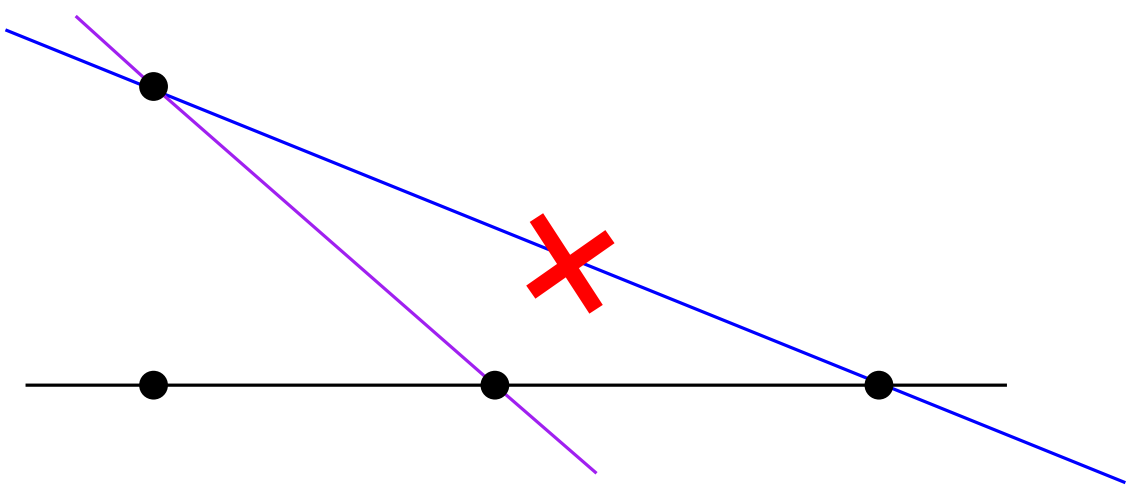

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There will be parameters (s, t) such that:

> each point is on $t + 1$ lines;
> each line is on $s + 1$ points

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For this, we will need the following:

- finite field, $GF(q)$, where q is even;
- projective space over $GF(q)$; and
- a hyperoval in projective plane over $GF(2^h)$.

Projective space

(Finite) projective space $PG(n, q)$ is an incidence structure in the $n+1$-dimensional space over $GF(q)$, where q is a prime power, where:

points: 1-dimensional subspaces

lines: 2-dimensional subspaces

planes: 3-dimensional subspaces

$\vdots$

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Some $f(t)$ which work:

t^2, t^6, t^{2^k} where k, h are coprime

The construction

$PG(3, 2^h)$

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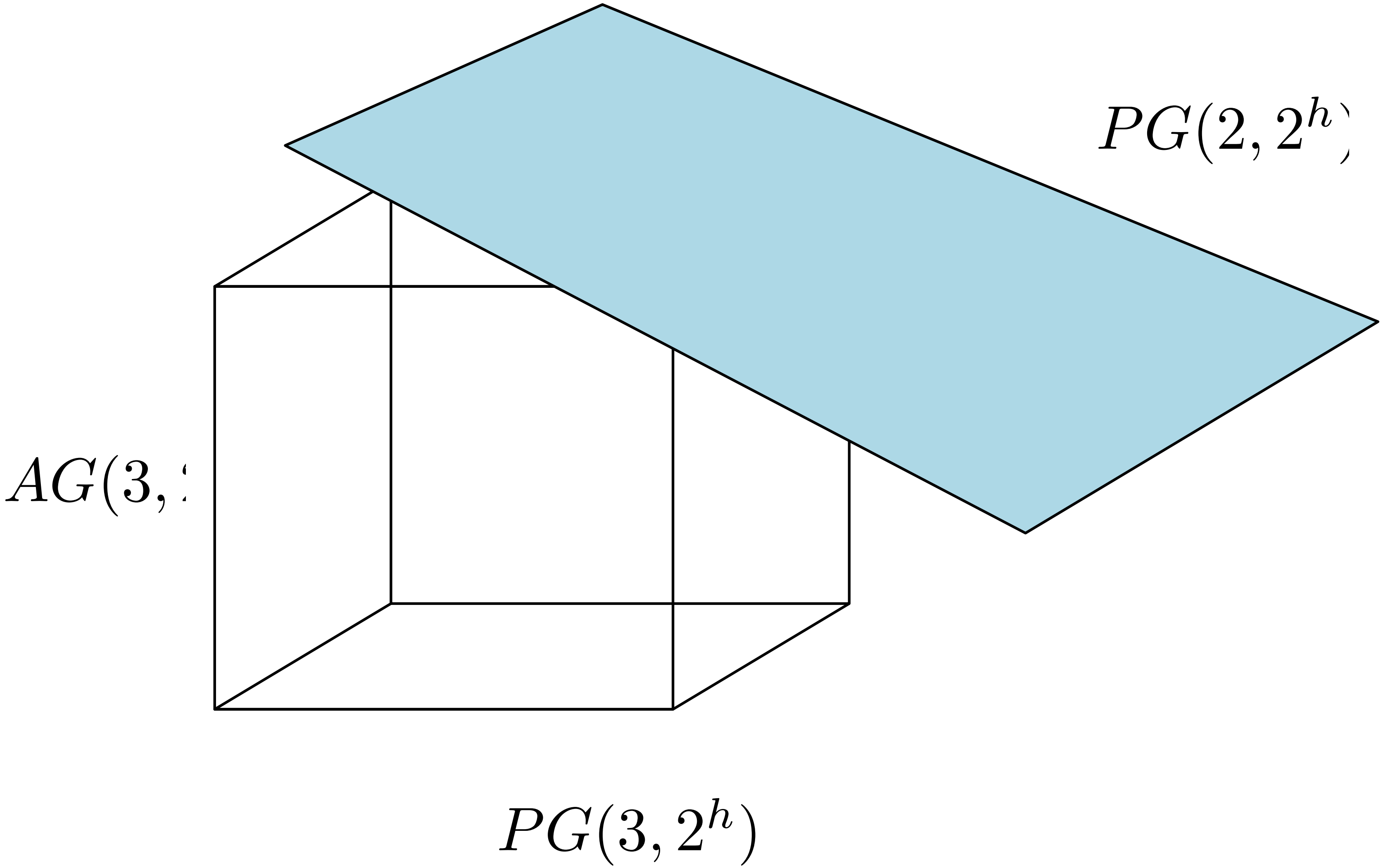

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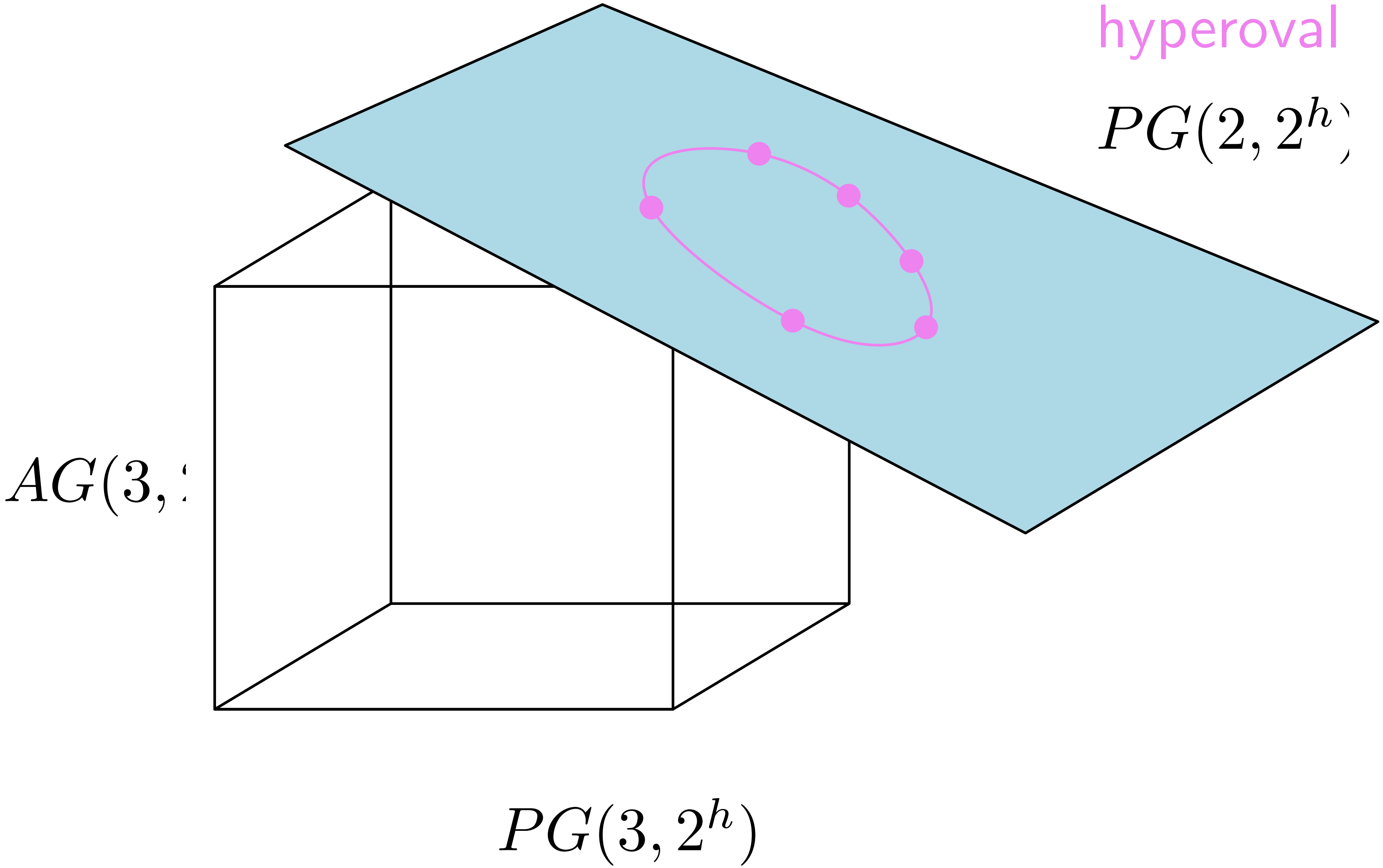

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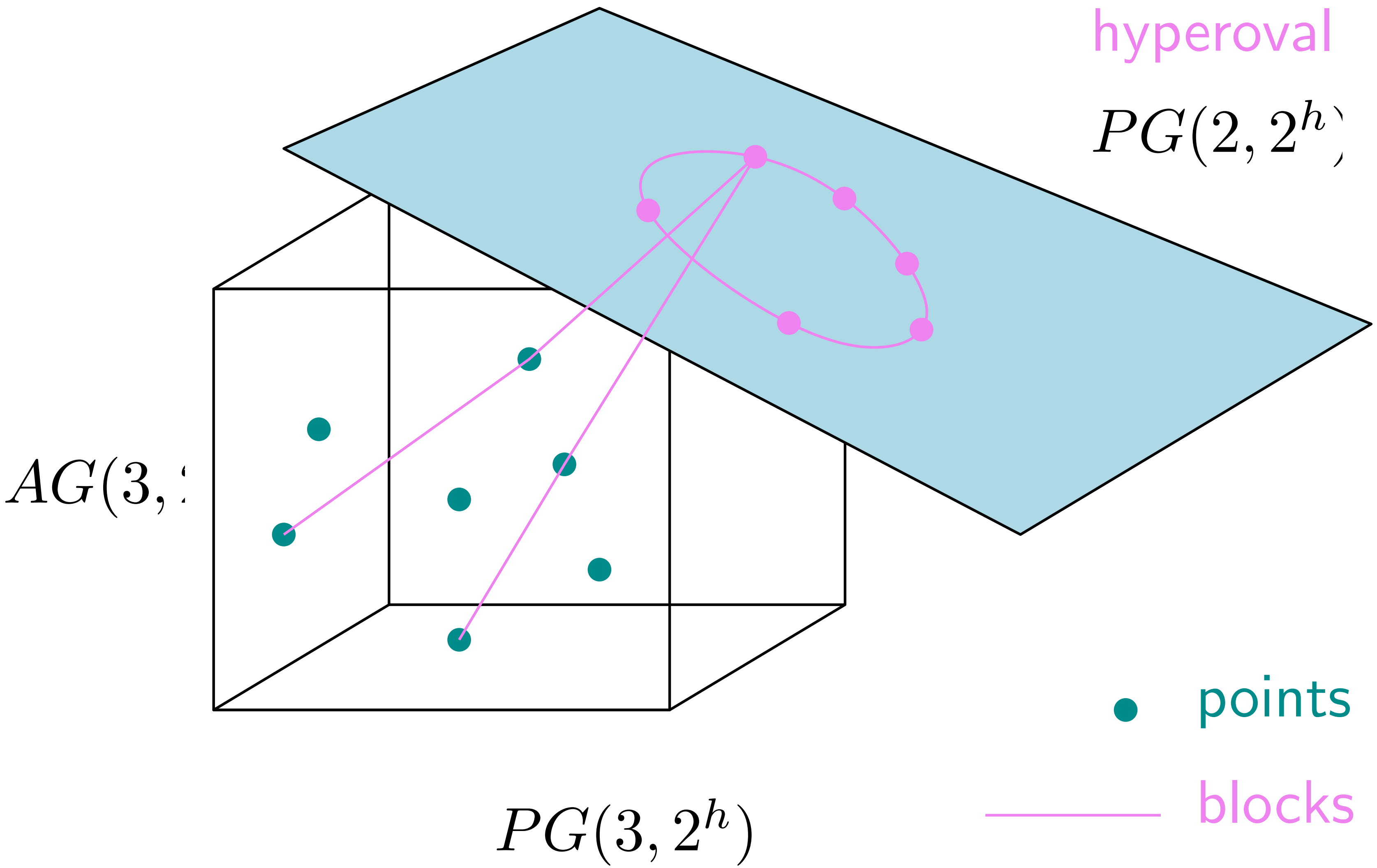

This will give a generalized quadrangle of order $(q-1, q+1)$, where $q = 2^h$.

Give it a go!

```
sage: gq4 = GQTstar(GF(4),"hyperconic")
sage: h = pointsGraph(gq4)
```

Try some stuff in SageMath!

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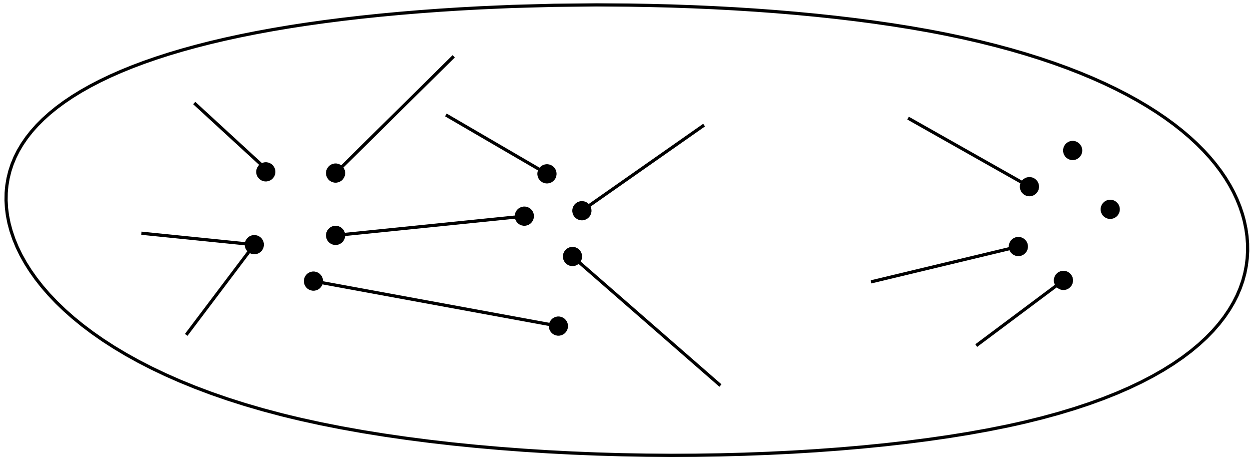

G

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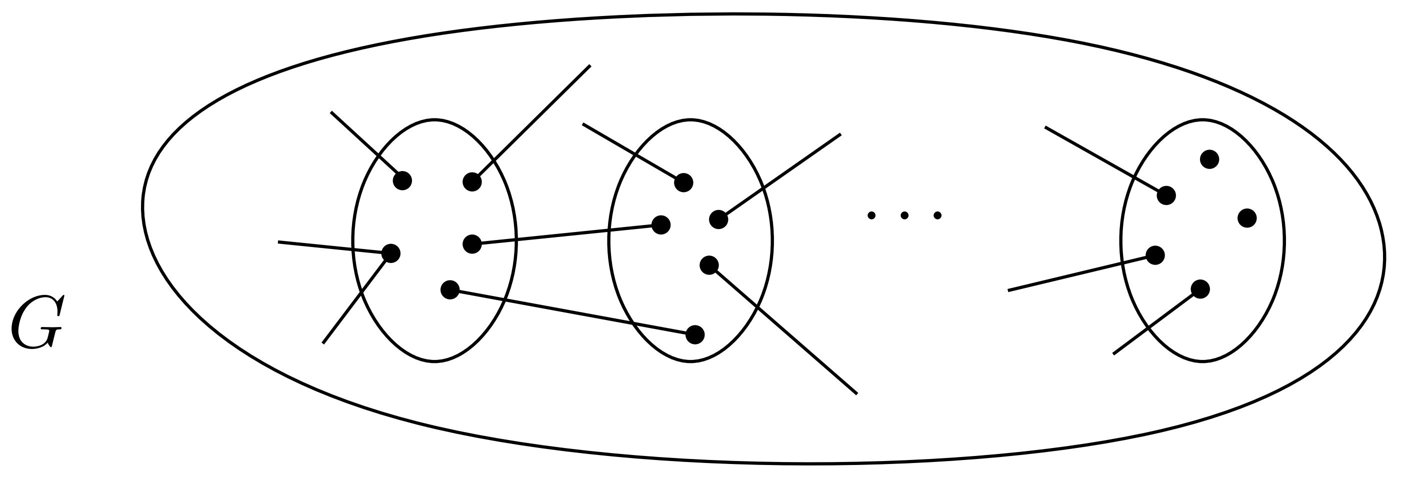

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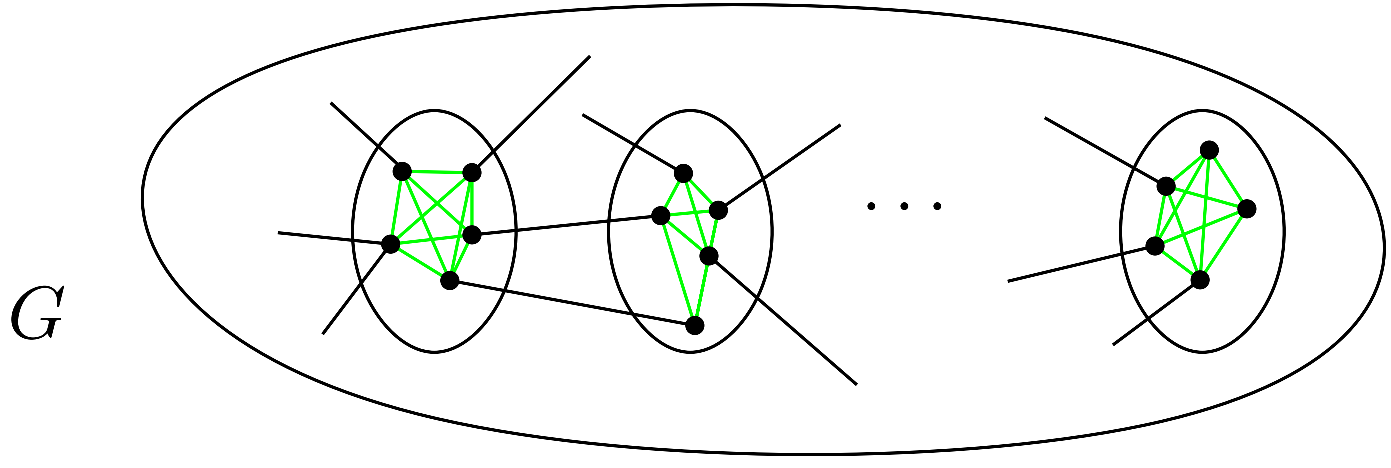
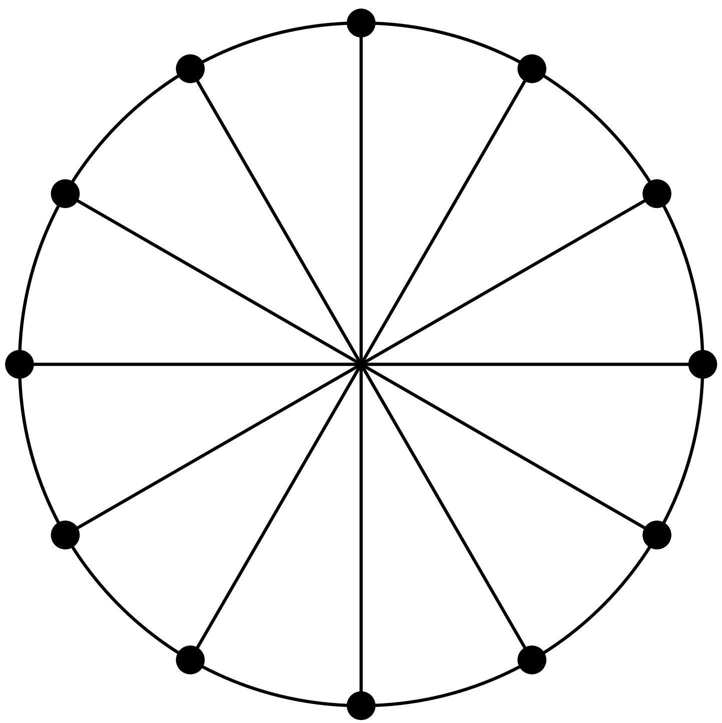
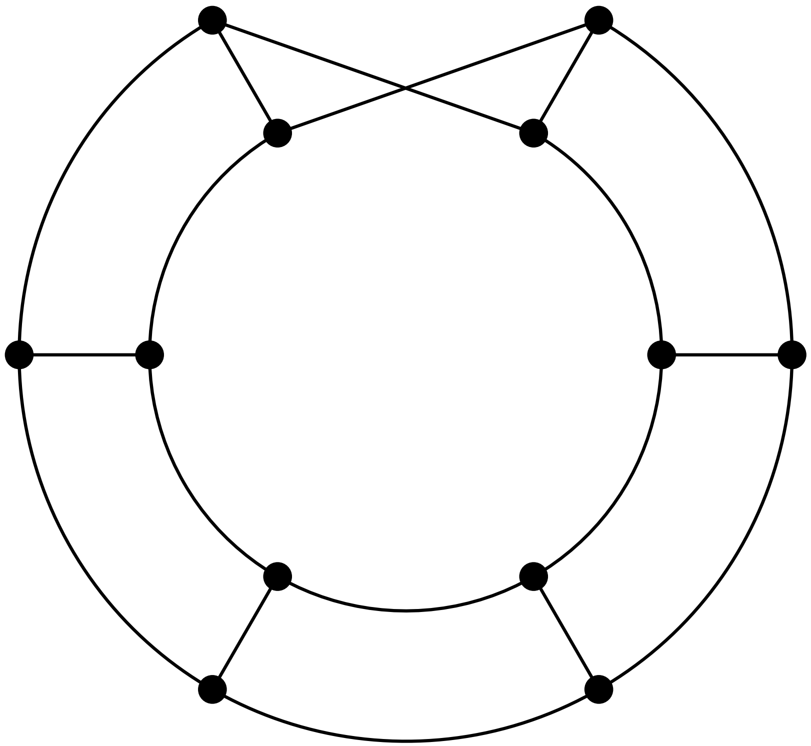


Or ...

Construct the following graphs. Are they isomorphic?





Installing sage

`https://doc.sagemath.org/html/en/installa`